THE MATHEMATICAL GAZETTE

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R. L. GOODSTEIN

WITH THE ASSISTANCE OF

H. M. CUNDY K. M. SOWDEN

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ELEMENTARY MECHANICS

by D. A. QUADLING, M.A., *Senior Mathematics Master, Marlborough College, and* A. R. D. RAMSAY, M.A., *Christ's College, Christchurch, New Zealand, formerly Senior Mathematics Master, Marlborough College. Volume I ready Autumn* 1957.

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There is a very large number of carefully graded examples to be worked by the student. In addition to the examples associated with each chapter there are four exercises containing in all 200 miscellaneous questions, taken mostly from examination papers.

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CLASSBOOK OF ARITHMETIC AND TRIGONOMETRY

by S. F. TRUSTRAM, M.Sc., *Principal, Salisbury and South Wilts College of Further Education, and* H. WHITTLESTONE, B.Sc., *Senior Mathematics Master, Grammar School, Normanton. Part I ready Autumn* 1957.

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The book will be issued complete and in two parts. The complete edition will cover fully the arithmetical topics and trigonometry required for the G.C.E. Ordinary level papers of the various examining boards.

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DR. H. MARTYN CUNDY, THE BEECHES, OBORNE ROAD, SHERBORNE, DORSET

MISS K. M. SOWDEN, NEWTON PARK COLLEGE, NEWTON ST. LOE, BATH

VOL. XLI MAY, 1957 NO. 336

EQUILIBRIUM OF SETS OF PARTICLES ON A SPHERE

BY JOHN LEECH

1. A number of particles are constrained to move freely on the surface of a sphere. Between every pair of them there acts a force which depends only on the distance between them, the law of force being the same for all particles. For a given law of force and a given number of particles, there are a number of configurations of the particles in which they are in equilibrium. In general, these configurations will depend on the law of force: we do not expect the particles to remain in equilibrium if the law of force is changed. But there are certain configurations in which the particles are in equilibrium whatever the law of force may be, for instance if the particles are at the vertices of a regular polyhedron. In this paper these configurations are investigated. It is evident that if a configuration has rotational symmetry about the diameter through a certain particle P of it, *i.e.* if the configuration can be rotated through a fraction of a revolution about the diameter through P and thereby brought into coincidence with its initial position, then the particle P is in equilibrium whatever the law of force may be. Thus among the configurations with the desired property are those which have this rotational symmetry about every diameter through particles of the system. It is shown that these are in fact the only such configurations. Specifically, the only such configurations are those in which the particles are equally spaced on a great circle, with or without two further particles one at each pole, and those which, in relation to a regular polyhedron,* comprise a set of particles at its vertices or a set at the mid-points of its edges or a set at the centres of its faces or any two or all three of these sets taken together. (This classification is redundant but it seems the most convenient.) No discussion of the stability of the configurations is given, as this depends in any given case on the law of force, the present discussion being confined to the property of equilibrium with independence of the law of force. The main proof occupies §§ 2–4. In §5 there is discussed the problem of two or more sets of particles on the sphere which are to be in equilibrium whatever the laws of force may be between particles both

* Throughout this paper a polyhedron is referred to by its central projection on to a concentric sphere, so that its "edges" and "faces" are great circle arcs and spherical polygons respectively.

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of the same set or of different sets. §6 relates the present problem to that of maximising the least distance between any two of an assigned number of particles.

2. Suppose we have a set of particles on the sphere which are in equilibrium whatever the law of force may be. Let us consider in particular a law of force by which the force between two particles is zero unless they are at a certain distance d apart. Each particle is then in equilibrium under the action of equal forces directed towards those particles distant d from it. When the particles distant d from a particle P are such that this is the case, we say that these particles are *balanced* about P. When a particle P is such that the particles at each distance d from P at which there are particles, are balanced about P, we say that the configuration is balanced about P, and when a configuration is balanced about every particle of it, we say that the whole configuration is balanced. Clearly then a configuration is in equilibrium under any law of force if and only if it is balanced. It is not necessary for there to be rotational symmetry about the diameter through a particle P for the configuration to be balanced about P; thus, if P is at the pole of a great circle and there are five particles on the great circle, three of which are equally spaced on it and the other two diametrically opposed but otherwise arbitrarily placed relatively to the first three, then these five particles are balanced about P although there is no rotational symmetry about the diameter through P. We show however that the whole configuration is balanced if and only if there is rotational symmetry about every diameter through particles of the configuration.

The following deductions about particles distant d from a particle P are immediate :

2.1. One particle is balanced about P only if it is antipodal to P.

2.2. Two particles are balanced about P if and only if they lie on a great circle through P.

2.3. If three particles are balanced about P, their joins* to P make angles of 120° at P.

2.4. If four particles are balanced about P, they lie two on each of two great circles through P.

These results are true for any distance d which occurs between pairs of particles of the configuration. Let us now consider the case of a law of force by which the force is zero except at the least distance δ which occurs between any pair of the particles. Let us call the particles at this least distance δ from

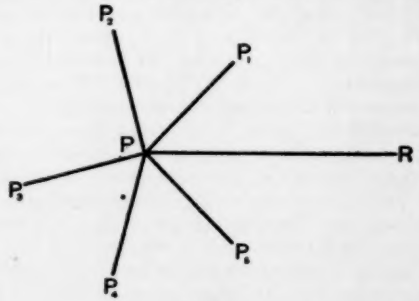

FIG. 1.

a particle its *neighbours*, and let particles which have no neighbours be called *isolated* particles. The following deductions about the neighbours of a particle P are immediate :

* Throughout this paper the term " join " means " great circle join ".

2.5. The joins of P to its neighbours make angles at P greater than 60°.

2.6. P has at most five neighbours.

2.7. If the joins of P to two of its neighbours are inclined at 120°, then P has precisely three neighbours.

2.8. If two neighbours of P lie on a great circle through P, then P has two or four neighbours.

2.9. If P has five neighbours, then the joins of P to its neighbours, taken consecutively in rotation, make angles less than 90°.

Of these the last alone needs explanation. Let the neighbours of P be $P_1\ P_2\ P_3\ P_4\ P_5$ in rotation, and suppose if possible that $\angle P_5PP_1 \geqslant 90°$. Let the tangential component of the force along PP_i be p, supposed attractive. Then since $\angle P_iPP_{i+1} > 60°$, there is a resultant force on P whose component in the tangential direction of RP, the bisector of $\angle P_5PP_1$, is of magnitude

$$\begin{aligned} &-p\Sigma \cos RPP_i \\ &> -p(2\cos 45° + 2\cos 105° + \cos 165°) \\ &= 0{\cdot}069p. \end{aligned}$$

This contradicts the requirement that the neighbours of P be balanced about P, and so the assumption $\angle P_5PP_1 \geqslant 90°$ is false.

3. We are now in a position to prove the fundamental

LEMMA : *The neighbours of a particle of a balanced configuration have equal numbers of neighbours,*

from which the final results follow almost at once. Separate proofs are necessary for the various cases (m, n) which arise when two neighbouring particles P, Q have m, n neighbours respectively. Except in the case (1, 1) of two antipodal particles, we may take $2 \leqslant n \leqslant m \leqslant 5$. For uniformity in notation, we denote by $Q, P_1, \ldots, P_{m-1}$ the neighbours of P in anticlockwise order, and by $P, Q_1, \ldots, Q_{n-1}$ the neighbours of Q in anticlockwise order. To save space, the argument is not given in full in some cases. There is no difficulty in completing these proofs, though some would be lengthy, the proofs given in full being quite typical. It should be noted that the diagrams are stereographic projections on to a plane tangent at P with consequent distortion of lengths. The only joins shown are between pairs of neighbouring particles, these represent the great circle joins and their angles of intersection are preserved.

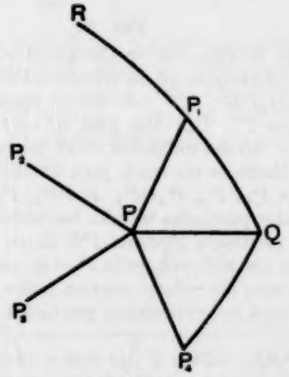

FIG. 2.

Case (5,5). Since $\angle P_1PQ < 90°$ and $\angle P_1QQ_4 < 90°$, $P_1Q_4 < \delta$ *i.e.* P_1 and Q_4 are the same particle. Similarly P_4 and Q_1 are the same particle. Since $\angle PP_1Q < 90°$, P_1 has at least four neighbours. Suppose, if possible, that it has just four neighbours. Then one of them, R say, lies on the great circle

through Q, P_1. Let us consider the balance about P of particles distant PR from it. Since $\angle P_1PR + \angle P_1RP > \angle PP_1Q = \angle P_1PQ$, $\angle P_1PR > \frac{1}{2}\angle P_1PQ$. Similarly the join of any particle distant PR from P is inclined to each join of P to its neighbours at an angle greater than $\frac{1}{2}\angle P_1PQ$. Consequently no join of P to a particle distant PR from it lies within either angle P_1PQ or QPP_4, or within $\frac{1}{2}\angle P_1PQ$ of PP_4. Thus there is no particle distant PR from P which lies on an arc of a circle whose pole is P, whose anticlockwise extremity is R and which contains an angle at P greater than 180°. It is therefore impossible for the particles distant PR from P to be balanced about P. The hypothesis that P_1 has four neighbours is therefore false and so it must have five. Similarly each other neighbour of P has five neighbours.

Case (5,4). Either $\angle PQQ_1 < 90°$ or $\angle PQQ_3 \leqslant 90°$, suppose the latter. Since $\angle QPP_1 < 90°$, $P_1Q_3 < \delta$ and so P_1 and Q_3 are the same. Thus $P_1Q = \delta$ and $\angle PP_1Q < 90°$, so that P_1 has at least four neighbours. Now if any neighbour of P had five neighbours, all would have, by case (5,5), hence P_1 has four

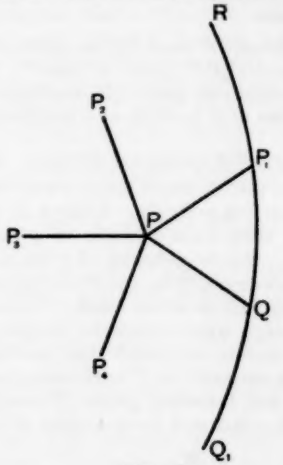

FIG. 3.

neighbours, one of which, R say, lies on the great circle through Q, P_1. Now Q_1, R are not antipodal (if they were, it would follow at once that P had only four neighbours) and $\angle Q_1PR > 120°$, so there must be at least two further particles distant PR from P. But the join of any one such particle to P is inclined to each join of P to its neighbours at an angle of at least $\angle P_1PR > \frac{1}{2}\angle QPP_1 > 30°$, whence there is no such join in the angle QPP_1 and at most one in each of the angles P_1PP_2, P_2PP_3, P_3PP_4, P_4PQ (for if there were two in any of these angles, the particles would be within δ of each other). Thus there can only be four particles distant PR from P, and this possibility is excluded by the fact that the reflections in P of Q_1 and R are within δ of P_2 and P_4 respectively. This case therefore cannot arise, *i.e.* there is no balanced configuration in which two neighbouring particles have five and four neighbours respectively.

Case (5,3). Since $\angle PQQ_2 = 120°$, P_1 is not a neighbour of Q, and $QP_1 > \delta$. Let us consider the balance about Q of the particles distant QP_1 from Q. There is at most one in each of the three angles at Q made by joins to its neighbours, and since the reflection of P_1 in Q is within δ of Q_1, the total number cannot be less than three and so is exactly three. One of these is within δ of P_4 and must therefore be P_4, so that the joins of Q to these three particles are the bisectors of the angles PQQ_1, Q_1QQ_2, Q_2QP at Q. Thus P_1, P_4 are also

neighbours of Q_2, Q_1 respectively, and the third particle, R say, is a neighbour of Q_1 and Q_2. Since $\angle PP_1Q_2 = \angle PQQ_2 = 120°$, P_1 has just three neighbours,

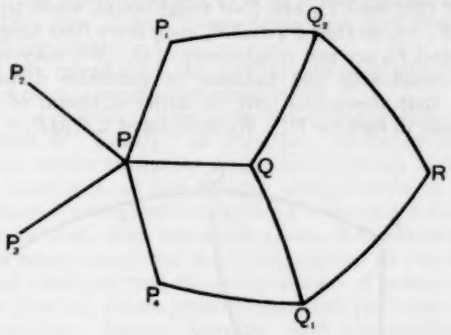

FIG. 4.

and so similarly have P_2, P_3, P_4. Among the neighbours of Q_2 are P_1, R, and so, since $\angle P_1Q_2R = \angle P_1PP_4$, Q_2 has five neighbours. So similarly has Q_1.

Case (5,2). Considering the balance about Q of particles distant QP_1 from Q, we find that there can only be two or four of them according as QP_1 and

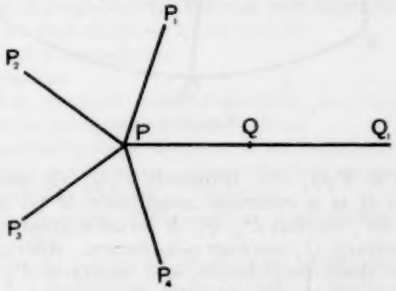

FIG. 5.

QP_3 are unequal or equal, and in the former case we find similarly that there are just two particles distant QP_4 from Q. Thus in either case there are particles which are the reflections in Q of P_1 and P_4. Thus Q_1 has five neighbours, two of them being these reflections. Since Q has two neighbours, no neighbour of P has more than two neighbours by the previous cases, and so they each have just two neighbours.

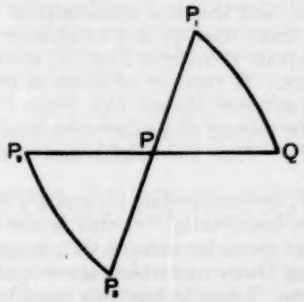

FIG. 6.

Case (4,4). Either $\angle P_1PQ + \angle Q_2QP \leqslant 180°$ or $\angle P_2PQ + \angle Q_1QP < 180°$, suppose the former. Then $P_1Q_2 < \delta$, and P_1 and Q_2 are the same. Consequently $\angle PP_1Q < 120°$ and P_1 has four neighbours, since by case (5,4) it has not five. Also $P_2P_3 = \delta$, so that P_2 and P_3 each have four neighbours similarly.

Case (4,3). P_1 and P_3 are not neighbours of Q. We may suppose $\angle P_1PQ \leqslant 90° \leqslant \angle QPP_3$. Considering the balance of particles distant QP_1 from Q about Q, we find that there can only be three of them, of which one, being within δ of P_3, must in fact be P_3. We now have $\angle P_1QP_3 = 120°$, so that QP_1

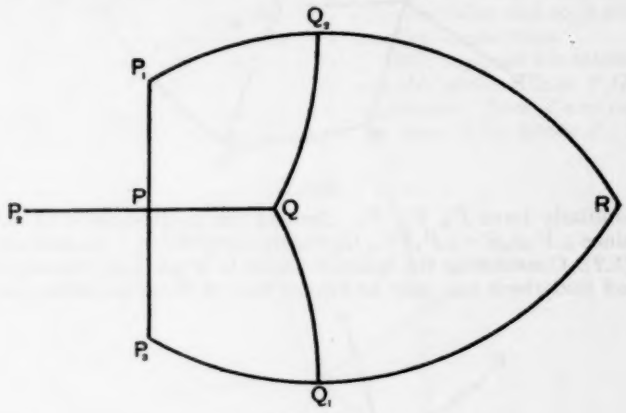

FIG. 7.

bisects $\angle PQQ_2$, and so $P_1Q_1 = \delta$. Similarly $P_3Q_1 = \delta$, and the third particle R distant QP_1 from Q is a common neighbour of Q_1 and Q_2. $\angle QQ_2R = \angle QQ_2P_1 = \angle QPP_1 = 90°$, so that P_1, Q_2, R lie on a great circle, and so Q_2 has four neighbours. Similarly Q_1 has four neighbours. Also $\angle PP_1Q_2 = \angle PQQ_2 = 120°$, so that P_1 has three neighbours, and similarly P_2 and P_3 have three neighbours.

Case (4,2). As in case (5,2) we find that among the neighbours of Q_1 are two which are the reflections in Q of two of the neighbours of P. In this case, these neighbours lie on a great circle through P and so their reflections lie on a great circle through Q_1, which therefore has four neighbours. Also since Q has two neighbours, no other neighbour of P has more than two neighbours by the previous cases, and so each has two.

Case (3,3). If Q and P_1 are neighbouring particles, each of P, Q, P_1, P_2 is a neighbour of each other, and this is a statement of the result. Again, if Q_2 is a neighbour of P_1, we have that Q_1 is a neighbour of P_2, and that P_1 and P_2 have a common neighbour R distinct from P, so that each neighbour of P has just three neighbours. If neither of these is the case, we find by considering the balance of particles distant PQ_1 from P about P and about P_2 and P_3 that there must be among these particles four, two of which are neighbours of P_2 and two of P_3. Thus the neighbours of P each have three neighbours.

Case (3,2). If P and Q_1 are antipodal, P_1 and P_2 are neighbours of both P and Q_1, and the result is immediate. If this is not the case, it is found by considering the balance of particles distant QP_1 from Q about Q and about Q_1 that there must be among them two which are neighbours of Q_1 which therefore has three neighbours. Since Q has two neighbours, no neighbour of P has more than two neighbours by the previous cases, and so all have two.

Case (2,2). Since Q has two neighbours, P and Q are not antipodal; nor therefore are P and P_1, so that P_1 has at least two neighbours. It has not more than two by the previous cases, and so it has just two.

Case (1,1). In this case P and Q are antipodal, and there are no further particles in the configuration.

This completes the proof that the neighbours of a particle of a balanced configuration have equal numbers of neighbours.

4. We are now in a position to consider the configurations on the sphere in their entirety instead of " locally " as hitherto. As the proofs for the various cases of the lemma nowhere assume that there are no isolated particles, it follows that the conclusion of the lemma is applicable to the non-isolated particles of a balanced configuration without reference to any isolated particles. We now show that, with one reservation, the conclusion of the lemma is sufficient as well as necessary for the configuration of non-isolated particles to be balanced, and conclude that the complete set of balanced configurations can be found by adjoining, where possible, isolated particles to configurations of non-isolated particles. Let us consider then the configurations of non-isolated particles on the sphere. Except in the case (1,1) of two antipodal particles, we may form a network on the sphere by joining all pairs of neighbouring particles. Since the angles between adjacent joins do not exceed 180°, the network is connected. When $m \neq n$, alternate particles in any path of the network have m and n neighbours, while when $m = n$, all the particles have equal numbers of neighbours. We can now construct the configurations on the sphere, and we find them to be as under:

(5,5). Vertices of an icosahedron.

(5,4). No configuration.

(5,3). Vertices of a rhombic triacontahedron, *i.e.* vertices and centres of faces of an icosahedron or regular dodecahedron.

(5,2). Vertices and mid-points of edges of an icosahedron.

(4,4). Vertices of an octahedron, cuboctahedron or icosidodecahedron, *i.e.* mid-points of edges of any regular polyhedron.

(4,3). Vertices of a rhombic dodecahedron, *i.e.* vertices and centres of faces of an octahedron or cube.

(4,2). Vertices and mid-points of edges of an octahedron.*

(3,3). Vertices of a tetrahedron, cube or regular dodecahedron.

(3,2). Vertices and mid-points of edges of a tetrahedron, cube or regular dodecahedron, or three particles equally spaced on a great circle with a particle at each pole.

(2,2). Three or more particles equally spaced on a great circle.

(1,1). Two antipodal particles.

This exhausts the configurations satisfying the conclusion of the lemma, all of which are balanced. Those balanced configurations with isolated particles are to be found by adjoining isolated particles where possible to these configurations. It is found that the only configurations into which it is possible to introduce isolated particles are those of particles at the vertices and mid-points of edges of a cube or regular dodecahedron and five or more particles equally spaced on a great circle. In these cases the only balanced configurations which can be derived are obtained by adjoining particles one at the centre of each face in the first two cases and one at each pole of the great circle in the last case.

To summarise these results, the balanced configurations may be divided into two families, an infinite family and a finite family. The infinite family

* In this case the configurations of particles at the vertices and mid-points of edges of a cuboctahedron and icosidodecahedron satisfy the conclusion of the lemma. They do not, however, satisfy the subsidiary result found in the proof of the lemma for this case that the neighbours of P and Q_1 be reflections in Q.

comprises the configurations of particles equally spaced on a great circle, with or without two further particles, one at each pole. The finite family comprises configurations having polyhedral symmetry, which are tabulated below. (The octahedral configuration is admitted to both families.) Except that 12′ and 20′ are not the vertices of the regular polyhedra, the occurrence of the same number in different places indicates different descriptions of the same configuration.

TABLE

listing the numbers of particles in balanced configurations having polyhedral symmetry in relation to the regular polyhedra.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
Polyhedron	Vertices	Centres of faces	Mid-points of edges	(i) & (ii)	(i) & (iii)	(ii) & (iii)	(i), (ii) & (iii)
Tetrahedron	4	4	6	8	10	10	14
Octahedron	6	8	12′	14	18	20′	26
Cube	8	6	12′	14	20′	18	26
Icosahedron	12	20	30	32	42	50	62
Dodecahedron	20	12	30	32	50	42	62

Note that each polyhedron is referred to by its central projection on to a concentric sphere, so that its edges and faces are great circle arcs and spherical polygons respectively.

5. The present problem may be generalised to that of finding configurations of two or more sets of particles which are in equilibrium whatever the laws of force may be between particles of any one set and whatever else it may be between particles of different sets. The solutions follow at once from those found above for a single set of particles. Each set of particles must itself form a balanced configuration, as there may be a null law between the particles of this set and of each other set, and any two or more of the sets taken together must form a balanced configuration, since the laws of force between the particles may be the same for particles both of any one set and for particles of different sets. In particular, the configuration of all the sets of particles taken together must itself be balanced. The problem is therefore resolved into that of finding balanced sets of particles having balanced subsets which when removed leave balanced sets. It follows at once that there can be at most three sets of particles. If there are two sets, they may be any two disjoint sets from the same line of the table, two sets of alternate particles on a great circle, or a set of particles on a great circle and the pair one at each pole. If there are three sets, they may be the sets in columns (i), (ii), (iii) of the table from the same line, or two sets of alternate particles on a great circle, and the pair one at each pole. A special case of this last is that of the three pairs of opposite vertices of an octahedron.

6. A problem of related interest is that of finding those configurations of n particles which maximise the least distance between any pair of them. Of configurations of particles which are in equilibrium under an inverse power of distance law of force, there are one or more which have minimum potential energy. As the index of the inverse power increases without limit, the only pairs of particles which contribute significantly to the potential energy are those at the least distance apart, and the configurations will approach (perhaps not continuously through all values of the index) the configurations which maximise the least distance between pairs of particles. Similarly any equilib-

rium configuration will approach a configuration for which the least distance between pairs of particles is stationary. As a balanced configuration is in equilibrium under any law of force, it is stationary for the least distance between pairs of particles. The question then arises of finding which balanced configurations maximise the least distance between pairs of particles.

It should be noted that the configurations obtained as limits in this way are not unique in regard to making stationary the least distance between pairs of particles. Thus the limiting process assigns exact positions for isolated particles whose positions are indefinite as far as the least distance between pairs of particles is concerned, and in the case of five particles, three on a great circle and one at each pole, the limiting process assigns equal spacing to the three particles on the great circle, while their actual spacing may be arbitrary as long as no two are less than 90° apart without affecting the least distance between pairs of particles. It may be noted also that a configuration which maximises the least distance between pairs of particles will not in general be in equilibrium under a law of force by which the force is zero except at the least distance between particles. A counter-example is the set of five particles at five of the vertices of an octahedron.

The only balanced configurations which maximise the least distance between pairs of particles are those of two antipodal particles, three particles equally spaced on a great circle, four particles at the vertices of a regular tetrahedron, five particles three on a great circle and one at each pole, six particles at the vertices of an octahedron, and twelve particles at the vertices of an icosahedron. Except for the case of five particles mentioned above, these configurations are unique. For the other balanced configurations it is possible to construct by deformation or otherwise examples showing that the balanced configurations are not maximal. These examples are given below.

(2,2). Four or more particles on a great circle, five or more if there is one at each pole. Displace alternate particles perpendicularly to and on opposite sides of the great circle, leaving one unmoved if the number is odd.

(5,2), (4,2), (3,2) except the case of five particles. Displace the particles at the mid-points of the edges perpendicularly to those edges.

(3,3). The regular tetrahedron is maximal. The cube can be deformed, by rotating one face relatively to that opposite it and then enlarging each of these two faces, into a square antiprism which is maximal. With the dodecahedron, rotate two opposite faces through 36° relatively to the remaining particles, displace the particles of these faces away from their centres and displace the remaining particles away from the great circle midway between the rotated faces.

(4,4). The octahedron is maximal. The cuboctahedron, regarded as the set of mid-points of edges of an octahedron, can be deformed by displacing each particle along the edge of the octahedron in such a direction that the displacements along the sides of a face are cyclic. This displacement will distort the cuboctahedron into an icosahedron, which is maximal. The icosidodecahedron, although a local maximum, is not the absolute maximum for 30 points as the length of its side is 36° which is less than that for the 32 vertices of the rhombic triacontahedron, namely $\cos^{-1}\sqrt{(\frac{1}{3}+\frac{2}{15}\sqrt{5})} \doteqdot 37° 20'$.

(4,3), (5,3). With the rhombic dodecahedron and triacontahedron, choose a great circle whose poles are vertices at which three or five edges meet respectively. This great circle is perpendicular to the edges which cross it. Rotate the whole set of particles on one side of this great circle relatively to those on the other (this will stretch these edges without, for a sufficiently small rotation, introducing any joins as short), and displace all the particles except those at the poles slightly towards this great circle in such a manner as to stretch every other edge. In the case of the rhombic triacontahedron, this deformation yields a very small improvement to the estimate given by Schütte and van

der Waerden* for the size of the smallest sphere upon which 32 points can be placed with no two less than unit distance apart.

There is no obvious way of relating the present problems to other extremal problems such as minimising the greatest distance at which an arbitrary point can be placed from the nearest point of a configuration. In fact, since a configuration which is not balanced is out of equilibrium under almost all laws of force, it is not to be expected that any such configuration will be found to be of significance in respect to both an equilibrium problem and another extremal problem, or even under two different significant equilibrium problems. It would seem therefore that the balanced configurations of this paper are the only "unique configurations" on the sphere in the sense of Whyte† and that any other configuration will be of significance only in relation to the problem out of which it arose.

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GLEANINGS FAR AND NEAR

1877. One becomes immune to anything—even the Japanese. My own recipe was the mental exercise of trying to recall to mind the second movement of Bruch's Violin Concerto and the proof of Pythagoras' Theorem about the square on the hypotenuse of a right-angled triangle. For months the air was made sad with a plaintive approximation of Bruch, whilst thousands of square yards of the air-strip were defaced with the complex lines of Theorem 74 drawn with the blade of my changkol in its white dust. Regrettably, I never did manage to prove Mr. Pythagoras' statement and when I was one day arrested for drawing the plans of a wireless set on the ground (nothing for hours would induce the guard to believe that the hundreds of lines he saw were related to anything so innocuous as geometry), I devoted my whole mind to the less dangerous problem of recapturing the strains of Bruch. It is remarkable to recall how many of my friends did not know how to prove the Pythagoras Theorem and could not whistle to me the second movement of the violin concerto by Bruch.—Russell Braddon, *The Naked Island* (PAN-Books). [Per Mr. J. R. Gisbon.]

1878. "This success" (the winning of the Brakenbury Competition) "was accompanied by an absurd failure. I was ploughed in 'Smalls'.... My failure was in arithmetic, and neither then nor ever after have I been able to do anything with sums. My brain still reels before the simplest exercises of addition, subtraction, or multiplication. I share Walter Scott's Pet Marjorie's opinion of them."—J. G. Lockhart, *Cosmo Gordon Lang*, p. 23. [Per Rev. A. F. MacKenzie.]

* *Math. Annalen* 123 (1951) 96–124 (p. 124). This whole paper is concerned with the problem of finding the maximum least distance between any two of n points on a sphere.

† *American Math. Monthly* 59 (1952) 606–611. This paper has an excellent bibliography of the subject of configurations of points on spheres, to which the reader is referred for further references.

FORMS AND FUNCTIONS

By Hazel Perfect

1. *Introduction.*

The two essentially different concepts of "form" and "function" are sometimes confused in elementary text-books, and in this note I set out to stress one or two important *formal* results, in particular the formal interpretation of the binomial theorem for a negative integral exponent. It has been one specific problem, namely the establishing of the *Wronski* relations

$$a_1 - h_1 = 0,\ a_2 - a_1 h_1 + h_2 = 0,\ \ldots,\ a_k - a_{k-1}h_1 + \ldots + (-1)^k h_k = 0,\ \ldots$$

connecting the elementary symmetric functions a_r and the homogeneous product sums h_r, which has recently focused my attention on this subject and prompted me to write down a few observations.

2. *Forms.*

Consider the set S of all formal expressions

$$\frac{a_0 + a_1 X + a_2 X^2 + \ldots}{a_0' + a_1' X + a_2' X^2 + \ldots}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where the a_r, a_r' are numbers, and not every a_r' is zero. If $a_r = 0$ or $a_r' = 0$ we find it convenient to omit the term $a_r X^r$ or $a_r' X^r$ in the expression. If $a_r = 0$ for every $r > n$ we write the expression as

$$\frac{a_0 + a_1 X + \ldots + a_n X^n}{a_0' + a_1' X + \ldots},$$

and similarly if $a_r' = 0$ for all $r > n$. If the denominator of the expression is the number 1 then we write the expression simply as $a_0 + a_1 X + a_2 X^2 + \ldots$. We speak of the members of S as *forms in the indeterminate* X, and we define next equality of forms, addition and multiplication.

$a_0 + a_1 X + a_2 X^2 + \ldots = b_0 + b_1 X + b_2 X^2 + \ldots$ means that $a_r = b_r$ for all r.

$$(a_0 + a_1 X + a_2 X^2 + \ldots) + (b_0 + b_1 X + b_2 X^2 + \ldots) = (a_0 + b_0) + (a_1 + b_1) X + (a_2 + b_2) X^2 + \ldots .*$$

$$(a_0 + a_1 X + a_2 X^2 + \ldots)(b_0 + b_1 X + b_2 X^2 + \ldots) = a_0 b_0 + (a_0 b_1 + a_1 b_0) X + \ldots + (a_0 b_r + a_1 b_{r-1} + \ldots + a_r b_0) X^r + \ldots .$$

$$\frac{a_0 + a_1 X + \ldots}{a_0' + a_1' X + \ldots} = \frac{b_0 + b_1 X + \ldots}{b_0' + b_1' X + \ldots}$$

means that $(a_0 + a_1 X + \ldots)(b_0' + b_1' X + \ldots) = (a_0' + a_1' X + \ldots)(b_0 + b_1 X \ldots)$.

$$\frac{a_0 + a_1 X + \ldots}{a_0' + a_1' X + \ldots} + \frac{b_0 + b_1 X + \ldots}{b_0' + b_1' X + \ldots} = \frac{(a_0 + a_1 X + \ldots)(b_0' + b_1' X + \ldots) + (a_0' + a_1' X + \ldots)(b_0 + b_1 X + \ldots)}{(a_0' + a_1' X + \ldots)(b_0' + b_1' X + \ldots)}.$$

$$\left(\frac{a_0 + a_1 X + \ldots}{a_0' + a_1' X + \ldots}\right)\left(\frac{b_0 + b_1 X + \ldots}{b_0' + b_1' X + \ldots}\right) = \frac{(a_0 + a_1 X + \ldots)(b_0 + b_1 X + \ldots)}{(a_0' + a_1' X \ldots)(b_0' + b_1' X + \ldots)}.$$

Write $$a(X) = \frac{a_0 + a_1 X + \ldots}{a_0' + a'_1 X + \ldots},$$

* The two uses of the sign + though at first sight unfortunate will not cause confusion.

and similarly, etc., and observe that if $a(X)=c(X)$ and $b(X)=d(X)$ then $a(X)+b(X)=c(X)+d(X)$ and $a(X)\,.\,b(X)=c(X)\,.\,d(X)$.

Addition and multiplication of forms have the usual familiar commutative, associative and distributive properties, and S contains as a subset the set of ordinary numbers.

It is convenient to denote by $a(X)^{-1}$ the form

$$\frac{a_0'+a_1'X+\dots}{a_0+a_1X+\dots},$$

and to use the usual notation for positive and negative integral powers.

We call those members of S of the form $a_0+a_1X+a_2X^2+\dots$ *formal power series*, those of the form $a_0+a_1X+\dots+a_nX^n$ *polynomials* (or more fully *polynomial forms*), and quotients of polynomials we call *rational forms*.

The fundamental result about polynomial forms is the *division algorithm*, which states that if $a(X)=a_0+a_1X+\dots a_nX^n$, and $b(X)=b_0+b_1X+\dots b_mX^m$, $(a_n\neq 0,\ b_m\neq 0)$, then there exist polynomials

$$q(X)=q_0+q_1X+\dots+q_{n-m}X^{n-m},\quad r(X)=r_0+r_1X+\dots+r_sX^s,$$

with $s<m$ and such that $a(X)=q(X)\,.\,b(X)+r(X)$. The result is well known and we omit the proof.

The notation (1) for members of S is the most suggestive, but of course we are simply considering ordered pairs of number sequences subject to certain prescribed rules of combination.

3. *The binomial theorem in S.*

$$(1+aX)^n=1+naX+\frac{n(n-1)}{2!}a^2X^2+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}a^rX^r+\dots, \quad (2)$$

for any integral n, positive or negative.

Proof. (i) Suppose n is a positive integer. In this case the series on the r.h.s. terminates, and we prove the result by induction on n.

For $n=1$ it is evidently true. Suppose

$$(1+aX)^{n-1}=1+(n-1)aX+\frac{(n-1)(n-2)}{2!}a^2X^2+\dots+\frac{(n-1)\dots(n-r)}{r!}a^rX^r+\dots.$$

Then
$$\begin{aligned}(1+aX)^n&=(1+aX)\left\{1+(n-1)aX+\dots+\frac{(n-1)(n-2)\dots(n-r)}{r!}a^rX^r\dots+)\right\}\\&=1+naX+\dots\\&\quad+\left\{\frac{(n-1)\dots(n-r)}{r!}+\frac{(n-1)\dots(n-r+1)}{(r-1)!}\right\}a^rX^r+\dots\\&=1+naX+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}a^rX^r+\dots;\end{aligned}$$

and this completes the proof by induction of case (i).

(ii) Suppose n is a negative integer. Write $n=-m$, then (2) becomes

$$\frac{1}{(1+aX)^m}=1-maX+\frac{m(m+1)}{2!}a^2X^2-\dots+(-1)^r\frac{m(m+1)\dots(m+r-1)}{r!}a^rX^r+\dots;$$

and we establish this by induction on m.

Now $(1+aX)\{1-aX+a^2X^2-\ldots+(-1)^r a^r X^r+\ldots\}=1$, whence the result is true for $m=1$. Suppose

$$\frac{1}{(1+aX)^{m-1}}=1-(m-1)aX+\frac{(m-1)m}{2!}a^2X^2-\ldots$$
$$+(-1)^r\frac{(m-1)m\ldots(m+r-2)}{r!}a^rX^r+\ldots.$$

Then
$$\frac{1}{(1+aX)^m}=\frac{1}{(1+aX)}\left\{1-(m-1)aX\right.$$
$$\left.+\frac{(m-1)}{2!}ma^2X^2-\ldots+(-1)^r\frac{(m-1)m\ldots(m+r-2)}{r!}a^rX^r+\ldots\right\}$$
$$=\{1-aX+\ldots+(-1)^r a^rX^r+\ldots\}$$
$$\times\left\{1-(m-1)aX+\ldots+(-1)^r\frac{(m-1)m\ldots(m+r-2)}{r!}a^rX^r+\ldots\right\}.$$

The coefficient of X^r on the r.h.s. is equal to

$$(-1)^r a^r({}^{m+n-2}C_r+{}^{m+n-2}C_{r-1}+\ldots+{}^{m-1}C_1+1)=(-1)^r a^r\,{}^{m+n-1}C_r;$$

and this completes the proof by induction of case (ii).

Note also in this connection the following theorem: For any numbers p, q

$$\left\{1+pX+\frac{p(p-1)}{2!}X^2+\ldots+\frac{p(p-1)\ldots(p-r+1)}{r!}X^r+\ldots\right\}$$
$$\times\left\{1+qX+\frac{q(q-1)}{2!}X^2+\ldots\right\}$$
$$=\left\{1+(p+q)X+\frac{(p+q)(p+q-1)}{2!}X^2+\ldots\right.$$
$$\left.+\frac{(p+q)(p+q-1)\ldots(p+q-r+1)}{r!}X^r+\ldots\right\}. \quad (3)$$

4. *Functions.*

Consider the rational form

$$\frac{a_0+a_1X+\ldots+a_nX^n}{a_0'+a_1'X+\ldots+a_m'X^m}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and suppose X is replaced by a number x (say). With $+$ interpreted as ordinary addition, juxtaposition signifying multiplication and the bar signifying division,

$$\frac{a_0+a_1x+\ldots+a_nx^n}{a_0'+a_1'x\ldots+a_m'x^m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

is itself a number y (say).* As x varies so in general does y, and we call y a *rational function* of x. We shall say that (5) is the *function* associated with the *form* (4). In particular polynomial functions are defined. Two rational functions may be said to be equal if they are equal for all x for which they are both defined. Evidently rational functions corresponding to equal rational forms are equal. The converse is less obvious, but is also true. It is a corollary of the result that if a polynomial function of degree n is equal to zero for $n+1$ distinct values of x then the coefficients are all equal to zero, which in turn is a consequence of the division algorithm.

* We exclude the case that $a'_0+a'_1x+\ldots+a'_mx^m=0$.

If x is a number which varies in such a way that $a_0+a_1x+a_2x^2+\dots$ is a convergent series then the series defines a function which is associated with the formal power series $a_0+a_1X+a_2X^2+\dots$; and similarly for quotients of convergent power series. We find it convenient to say here that two functions are equal if they are equal for all values of x for which they are both defined and which make the series appearing in the expressions for the functions absolutely convergent. With this definition, formal equality implies functional equality. The converse is also true and follows from the result that a function cannot be represented by two different power series.*

The binomial theorem for functions:

$$(1+ax)^n=1+nax+\frac{n(n-1)}{2!}a^2x^2+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}a^rx^r+\dots,$$

with n a positive or negative integer, follows from the theorem for forms provided we establish also (in the case $n<0$) the absolute convergence of the series on the r.h.s. for $|ax|<1$. Also, looking at it from the opposite point of view, if the binomial theorem has been established for functions, then this implies its truth for forms.

Formal results are sometimes established more simply using functional methods. For example, the result (3) of § 3 follows at once from the general binomial theorem for functions and the index law.

5. *A wider interpretation of the coefficients in a form.*

In a sense (made precise in § 4) formal equality and functional equality are equivalent when the coefficients are numbers. For this reason, although the concepts of form and function are very different, in practice we may pass from one interpretation to the other very easily, and we may tend to lose sight of the distinction. However, if we no longer assume that the coefficients are numbers, then the functional interpretation, in so far as it depends on the theory of convergence, breaks down, while the formal interpretation often remains essentially valid. For example, we might suppose that the coefficients are themselves indeterminates, or that they belong to some finite field. In this latter case, even for polynomials, functional equality no longer in general implies formal equality.†

In conclusion I turn to the problem of establishing the Wronski relations. We write $a_1=\Sigma x_1$, $a_2=\Sigma x_1x_2$, $\dots$, $a_n=x_1x_2\dots x_n$, and call $a_1, a_2, \dots, a_n$ the elementary symmetric functions of $x_1, x_2, \dots, x_n$.

The homogeneous product sum h_r is the sum of the products of $x_1, x_2, \dots, x_n$ taken r at a time and with unrestricted repetition.

Evidently $(-1)^ra_r$ is the coefficient of X^r in the form

$$(1-x_1X)(1-x_2X)\dots(1-x_nX),$$

and h_r is the coefficient of X^r in the form

$$(1+x_1X+x_1^2X^2+\dots)(1+x_2X+x_2^2X^2+\dots)\dots(1+x_nX+x_n^2X^2+\dots).$$

This second form is equal to $[(1-x_1X)(1-x_2X)\dots(1-x_nX)]^{-1}$; in other words, the product of the two forms is equal to 1, and the Wronski relations result from equating to zero the coefficients of $X, X^2, \dots$ in this product.

The nature of the quantities $x_1, x_2, \dots, x_n$ in the Wronski relations need not be specified, and it is the theory of forms and not functions which is required for the establishing of the relations.

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* See for example Hardy's "Pure Mathematics" pp. 290, 386. We must assume in addition that every power series which occurs has a positive radius of convergence.

† The two polynomial functions x^2-x+1, 1 with coefficients in the field consisting of the two elements 0, 1 are equal in this field but they are formally different.

GEODESIC OPPOSITES ON A REGULAR TETRAHEDRON

1. Consider a regular tetrahedron $ABCD$ of side p (Fig. 1); let P be a point on the face ABC and Q a point on the face BCD; we first consider the geodesic distance between P and Q.

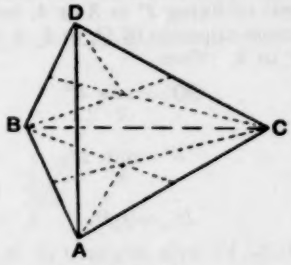


FIG. 1.

On unfolding the surface of the tetrahedron into the plane of ABC, the face BCD takes up one of five different positions, a, b, c, d, e, (Fig. 2). Divide ABC into regions 1, 2, 3, 4, 5, and BCD into regions 1′, 2′, 3′, 4′, 5′, as shown.

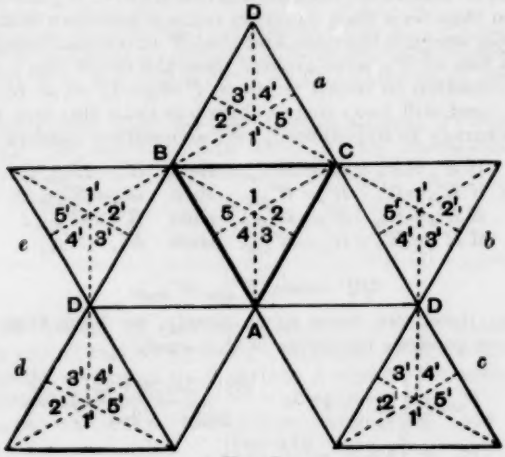


FIG. 2.

Let R_k be the straight line distance PQ_k, where Q_k is the position of Q in the position corresponding to k ($k = a, b, c, d, e$). Let R be the geodesic distance PQ; then

$$R = \min(R_a, R_b, R_c, R_d, R_e). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

2. We have immediately the following results; that if P is in m and Q is in n', then:

m or $n = 1$ $m = n = 2$ or 5	$R = R_a$,
$1 < m < n$	$R = \min(R_a, R_b)$,
$1 < n < m$	$R = \min(R_a, R_e)$,
$m = n = 3$	$R = \min(R_a, R_b, R_c, R_e)$,
$m = n = 4$	$R = \min(R_a, R_b, R_d, R_e)$.

3. *Geodesic Opposites.* Keeping P fixed, let P' be that point on the surface of the tetrahedron whose geodesic distance from P is a maximum; call P' the geodesic opposite of P. (It does *not* follow that P is the geodesic opposite of P'.)

From the results of para. 2, we have that if P is in 1, 2, or 5 then P' is not in BCD, and that if P is in 3 (or 4) then P' is in 3′ (or 4′).

For convenience, instead of fixing P in 3 or 4, we fix a point Q in 3′ or 4′, and look for Q', the geodesic opposite of Q, in 3 or 4. By symmetry we can take Q to be in 3′ and Q' in 3. Then

$$QQ' = \max_{\mathrm{P}} R',$$

where

$$\left.\begin{aligned} R' &= \min_{\mathrm{k}} R'_k, \\ R'_k &= Q_k P. \end{aligned}\right\} \quad (k = a, b, c, e)$$

and

4. Take coordinates $O(X, Y)$ with origin O at A, OX parallel to BC, and OY perpendicular to OX cutting BC in $(0, \frac{1}{2}\sqrt{3}\,p)$. Let $Q_e = (x, y)$. Then

$$Q_a = (-p-x,\ \sqrt{3}p-y);\quad Q_b = (2p+x,\ y);\quad Q_c = (-x,\ -y).$$

Let $P = (X, Y)$; we can write down ${R'_k}^2$ in terms of X, Y, x, y, for each k. As P varies in 3, R' attains its maximum when three of R'_k are equal; subject to the condition that their then common value is less than that of the fourth. (For suppose this were not the case, and that R' was a maximum for a position of P for which two of R'_k were greater than the other two; then we could always find a direction in which to move P slightly so as to increase these second two R'_k and still keep their values less than the first two R'_k, so increasing R'—contrary to hypothesis.) By elementary algebra we find that

$$\begin{aligned}
&\text{if } R'_a = R'_c = R'_e = R'_{ace}, \quad \text{then} \quad R'_b < R'_{ace};\\
&\text{if } R'_a = R'_b = R'_c = R'_{abc}, \quad \text{then} \quad R'_e < R'_{abc};\\
&\text{if } R'_a = R'_b = R'_e = R'_{abe}, \quad \text{then} \quad R'_c > R'_{abe};\\
&\text{if } R'_b = R'_c = R'_e = R'_{bce}, \quad \text{then} \quad R'_a > R'_{bce}.
\end{aligned}$$

Hence

$$QQ' = \max(R'_{abe},\ R'_{bce}).$$

5. Examining these two cases more closely, we have that there are two possible points as geodesic opposites of Q, namely

$$\left.\begin{aligned} P_{abe} &= \left\{p+x,\ \tfrac{1}{2}\sqrt{3}p - \frac{(3p+2x)(p+2x)}{2(2y-\sqrt{3}p)}\right\}, \\ P_{bce} &= \left\{p+x,\ -\frac{x(p+x)}{y}\right\}; \end{aligned}\right\} \qquad \text{.....(2)}$$

and that the corresponding geodesic distances QP_{abe}, QP_{bce}, are given by

$$\begin{aligned}
{R^2}_{abe} &= p^2 + \left\{\frac{\sqrt{3}p-2y}{2} + \frac{(3p+2x)(p+2x)}{2(\sqrt{3}p-2y)}\right\}^2,\\
{R^2}_{bce} &= p^2 + \left\{y + \frac{x(p+x)}{y}\right\}^2.
\end{aligned}$$

Now

$${R^2}_{abe} - {R^2}_{bce} = \frac{p(x+\sqrt{3}y)(\sqrt{3}p+\sqrt{3}x-y)}{2y^2(\sqrt{3}p-2y)^2}\quad F(x, y)$$

where

$$F(x, y) = y(2y-\sqrt{3}p)(4y-\sqrt{3}p) + y(3p+2x)(p+2x) + 2x(p+x)(2y-\sqrt{3}p);$$

hence if $F(x, y) > 0$, $QQ' = R_{abs}$,
and if $F(x, y) < 0$, $QQ' = R_{bcs}$;

and the subscripts give the three possible positions of Q_k from which the geodesic journey to Q' is made.

The two subregions $3'_1$ (DHK) and $3'_2$ (GHK) of $3'$ are separated by the curve $F(x, y) = 0$ (see Fig. 3). To $3'_1$ and $3'_2$ correspond subregions 3_1 and 3_2

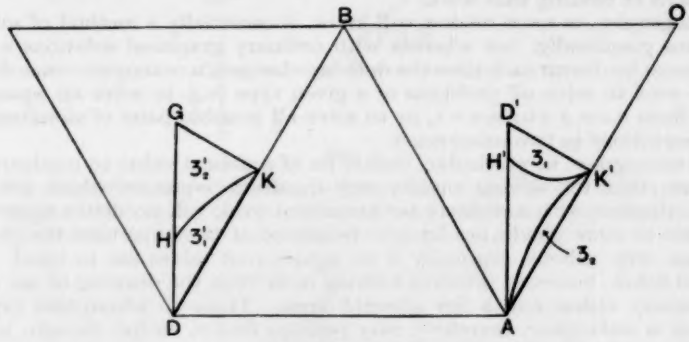


FIG. 3.

of 3; Q' lies in 3_1 or 3_2 according as Q lies in $3'_1$ or $3'_2$. (Note that there is a large region of 3 in which Q' cannot lie, whatever the position of Q.) The bounding curves of 3_1 and 3_2 are given by (2), where (x, y) are connected by $F(x, y)$.

It should be noted that in the correspondence $3'_1 \rightarrow 3_1$, $KH \rightarrow K'H'$, $HD \rightarrow H'D'$, and $DK \rightarrow D'K'$; and that in the correspondence $3'_2 \rightarrow 3_2$, $KG \rightarrow$line $K'A$, $KH \rightarrow$curve $K'A$, and $GH \rightarrow$point A.

Decca Research Laboratories H. G. APSIMON

1879. The law, passed at the last session of the legislature, and taking effect today, is aimed mainly at stock racketeers in Montreal who allegedly sold new worthless securities to American investors for several times their value.—*The Herald Tribune* (N.Y.)

Inexpensive, anyway.—*The New Yorker*, August 20, 1955. [Per Mr. H. E. Deming.]

1880. Practical results of a convincing—even spectacular—nature, provided by the study groups of members of the Work Study Appreciation Course, were a feature of the North Gloucestershire Productivity Association Work Study Week.

In one instance, where recommendations were immediately put into effect, a 200 per cent. reduction in labour time on a particular job was effected, and the girls concerned greatly preferred the new method to the old.—*British Productivity Council Bulletin*, September, 1955. [Per Mr. S. F. Trustram.]

1881. ARCHIMEDES. On hearing his name, shout "Eureka!" Or else: "Give me a fulcrum and I will move the world." There is also Archimedes' screw, but you are not expected to know what it is.

ASTRONOMY. Delightful science. Of use only to sailors. In speaking of it, make fun of astrology.

SOLVING QUADRATICS QUICKLY

By R. F. Wheeler

The writer wishes first to acknowledge his indebtedness to Prof. Brodetsky's book [1] on Nomography ; the following observations were, in fact, suggested as a result of reading that work.

Nomography, as most readers will know, is essentially a method of solving problems graphically, but whereas with ordinary graphical solutions a new curve must be drawn each time the data are changed, a nomogram once drawn can be used to solve all problems of a given type (e.g. to solve all equations of the form $a\cos x + b\sin x = 1$, or to solve all possible pairs of simultaneous linear equations in two unknowns).

One nomogram, in particular, should be of practical value to mathematics teachers—that for solving rapidly any quadratic equation which has real roots. Readers with a distaste for numerical work will no doubt agree that when six or more quadratics have to be solved at the same time the process becomes very tedious, especially if no square root tables are to hand. The method below, however, involves nothing more than the drawing of one circle of arbitrary radius and a few straight lines. Those to whom this type of solution is unfamiliar, therefore, may perhaps find it useful, though, in any case, the method is interesting for its own sake.

The basic purpose of a nomogram is to solve for x an equation of the type

$$f(x, a, b) = 0 \quad \ldots\ldots\ldots\

Take coordinates (ξ, η) in the plane of the nomogram. Without loss of generality, we may take the A-scale as the η-axis and its zero as the origin. Suppose that the B-scale makes an angle θ with the positive ξ-axis and that its zero

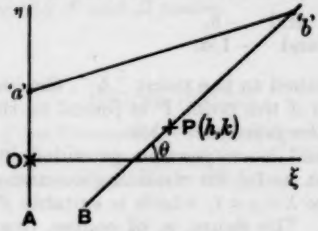


FIG. 1.

is at the point $P(h, k)$. Suppose, further, that A and B are both graduated linearly, the a-unit being λ times, and the b-unit μ times the unit of coordinates. Then the point "a" has coordinates $(0, a\lambda)$, the point "b" is

$$(h + b\mu \cos\theta,\ k + b\mu \sin\theta)$$

and the equation of the line joining "a" and "b" reduces to

$$ab\lambda\mu \cos\theta + a\lambda(h - \xi) + b\mu(\xi \sin\theta - \eta \cos\theta) + (k\xi - h\eta) = 0. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

The equation of X is found by making (2) identically equal to either (L), (M) or (N) for all values of a and b. Now each of the arrangements (L), (M), (N) is a linear expression in a and b, and so, whichever is chosen, (2) can contain no term in ab. Since, therefore, neither λ nor μ can be zero, $\cos\theta = 0$, and so A and B must be parallel. We may take $\theta = \frac{1}{2}\pi$; $\theta = \frac{3}{2}\pi$ is equivalent to graduating B in the opposite direction, and this can be allowed for, if necessary, by changing the sign of μ. Equation (2) then reduces to

$$a\lambda(h - \xi) + b\mu\xi + (k\xi - h\eta) = 0. \quad \ldots\ldots\ldots\ldots\ldots(3)$$

If we attempt to identify (3) with either (L) or (N), we find that it is impossible, on account of our assumption of linear scales, to make X a circle. (The reader will easily verify this for himself if he makes the necessary changes in the subsequent argument.) We shall see, however, that it *is* possible with (M), for if (3) and (M) are identical for all a and b, then

$$x^2 : x : 1 = \lambda(h - \xi) : (k\xi - h\eta) : \mu\xi,$$

which gives ξ and η parametrically in terms of x. Eliminating x, therefore, we obtain for the equation of X

$$(k^2 + \lambda\mu)\xi^2 - 2hk\xi\eta + h^2\eta^2 - \lambda\mu h\xi = 0. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Now (4) represents a circle if

$$hk = 0 \quad \text{and} \quad k^2 + \lambda\mu = h^2. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Obviously, $h = 0$ is inadmissible, since it removes the term in η^2. Therefore $k = 0$, and since we have not yet fixed the unit of coordinates, there is no loss of generality in taking $h = 1$, so that P is the point $(1, 0)$. Then, from (5),

$$\lambda\mu = 1, \quad \ldots\ldots\ldots\ldots\ldots(6)$$

and we note, in particular, that this implies that A and B must be graduated in the same direction. The X-curve becomes

$$\xi^2 + \eta^2 - \xi = 0,$$

which is the circle on OP as diameter.

If we wish to graduate the circle with the appropriate values of x, the simplest method is to observe that the equation $ax^2+x+b=0$ has certain special solutions:

1. If $a=0$, $x=\infty$ and $-b$.
2. If $b=0$, $x=0$ and $-1/a$.

Thus if the origin O is joined to the point "b", the line cuts the circle again at the point $x=-b$, and if the point P is joined to the point "a", the line cuts the circle again at the point $x=-1/a$.

Various scales for a and b are possible provided (6) is satisfied, and this freedom of choice is most useful for obtaining accurate roots.

Fig. (2) shows the case $\lambda=\mu=1$, which is suitable if the roots are numerically less than about 2. The figure is, of course, drawn very much smaller

FIG. 2.

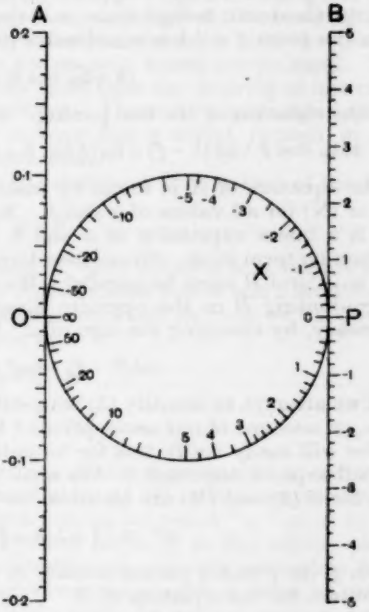


FIG. 3.

than it would be in practice—5 cm. is a convenient radius for most purposes. For roots numerically between, say, 2 and 10, Fig. 3 would be preferable; here $\lambda=5$, $\mu=0{\cdot}2$.

Of course, if the nomogram is being used only for casual solutions, as has been supposed, it is quite unnecessary to graduate the circle. If the line joining the points "a" and "b" cuts the circle at S and T and OS, OT cut the B-scale at U and V, then the graduations at U and V *with the signs changed* give the roots of the equation. Thus, to solve $5x^2-24x-61=0$, we arrange it as $ax^2+x+b=0$ with $a=-0{\cdot}208$, $b=+2{\cdot}54$. On joining these points on the A and B scales in, say, Fig. 3 (when drawn on a larger scale), we find that the points U and V referred to above are marked 1·84 and −6·64, so that the roots are −1·84 and +6·64. (The sum of the roots provides a check.)

It is a simple matter to choose the best values of λ and μ in a particular case, and this done, the roots of the quadratics are obtained with surprising ease, particularly since the circle will normally be drawn on graph paper which saves calibrating the A and B scales.

Hymers College, Hull. ROGER F. WHEELER

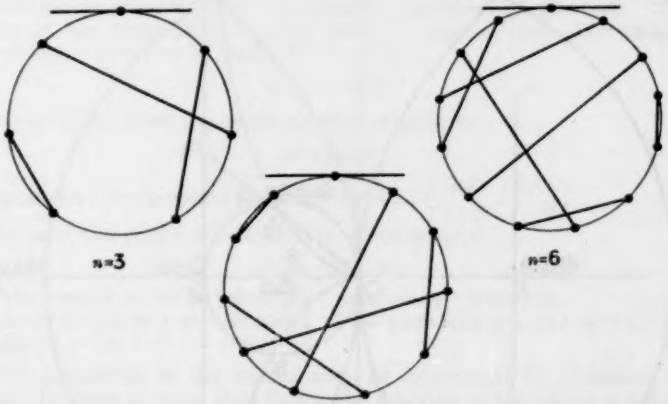
REFERENCES

1. S. Brodetsky, *A First Course in Nomography*. G. Bell & Sons.
2. P. M. d'Ocagne, *Traité de Nomographie*. G. Villars, Paris.
3. E. T. Whittaker, *Edin. Math. Soc.* Notes no. 19 (Dec. 1915), p. 125.

PROBLEM

The following problem arose in connection with the design of " movements " (i.e. circulations of players and cards) for Duplicate Bridge Competitions. It is presented here in geometrical form, as it is easiest to explain and to understand in this guise.

Let $(2n+1)$ points be plotted, equally spaced, around the circumference of a circle. Now let $2n$ of them be joined in pairs by means of n chords, and let the tangent at the remaining point be drawn. It is required, if possible, to do this in such a way that no two of the chords are equal in length, and that no chord is parallel either to any other chord or to the tangent at the last point.



Simple trial and error will show that these conditions cannot be satisfied for $n=1$, 2, or 4, but that constructions answering the conditions exist for $n=3$, 5, and 6. Rather more laborious trial and error will convince that there is no solution for $n=7$; but the problem can be shown to be soluble for 8, 9, and 10.

The question is of course this: what is the formal criterion which determines, for any value of n, whether or not the conditions can be satisfied?

The sketches above show solutions for 3, 5, and 6.

D. J. BEHRENS

THE CISSOID OF DIOCLES

By J. P. McCarthy.

1. The polar equation of the parabola $y^2 = 8ax$, referred to the origin of coordinates as pole is

$$r \sin^2 \theta = 8a \cos \theta$$

or
$$r = \frac{8a \cos \theta}{\sin^2 \theta}.$$

Inverting with the pole as centre of inversion and $4a$ as the radius of inversion we find that the equation to the inverse curve is

$$r = \frac{2a \sin^2 \theta}{\cos \theta}$$

or
$$r \cos \theta = 2a \sin^2 \theta$$

or
$$r = 2a \sin \theta \tan \theta.$$

The cartesian equation of the inverse curve is

$$y^2(2a - x) = x^3.$$

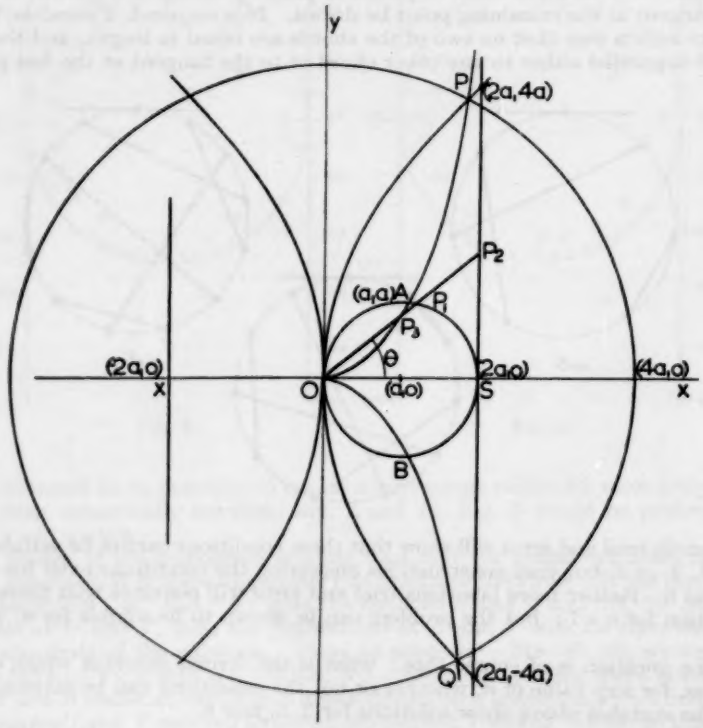


Fig. 1.

This curve is the Cissoid of Diocles.

2. (*a*) The latus rectum $x=2a$ of the parabola cuts the parabola at the points $(2a, \pm 4a)$ and is the asymptote to the cissoid.

(*b*) The parabola and the cissoid both pass through the origin, are both symmetrical about the axis of x and intersect at the points P and Q whose coordinates are $[4a(\sqrt{2}-1), \pm 4a\sqrt{2}(\sqrt{2}-1)]$.

(*c*) If O is the origin or pole we have $OP=OQ=4a$, so that P and Q are both on the circle of inversion, each inverting into itself.

(*d*) Thus the parabola, the cissoid and the circle of inversion all pass through P and Q.

3. (*a*) The circle $x^2+y^2-2ax=0$ passes through the origin and intersects the cissoid at the points $(a, \pm a)$. It also touches the parabola at the origin and the two have no other real common points. In fact the circle lies entirely within the parabola.

(*b*) If a straight line through the origin and making an angle θ with the positive direction of the x axis cuts this circle at P_1 and cuts the asymptote $x=2a$ at P_2 and the cissoid at P_3 we have

$$OP_2=\frac{2a}{\cos\theta},\ OP_1=2a\cos\theta \text{ and } OP_3=\frac{2a\sin^2\theta}{\cos\theta}$$

whence $$P_1P_2=OP_2-OP_1=\frac{2a\sin^2\theta}{\cos\theta}=OP_3.$$

Hence, as P_2 moves along the line, or P_1 around the circle, the point P_3 traces out the cissoid.

This property could be taken as a definition of the cissoid, whereby the curve is derived from the circle without using the parabola directly.

4. Now let us consider the "conjugate" parabola $y^2+8ax=0$. The coordinates of any point on this curve are $x=-2at^2$, $y=4at$. The tangent to the curve at $(-2at^2, 4at)$ is easily shown to be

$$x+ty=2at^2.$$

The perpendicular from the origin to this tangent is

$$tx-y=0.$$

The tangent and the perpendicular intersect at $\left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right)$.

The locus of the point of intersection is the cissoid

$$y^2(2a-x)=x^3,$$

so that the cissoid is the pedal of the "conjugate" parabola.

[It follows similarly that the pedal of the parabola $y^2=8ax$ is the "conjugate" cissoid $y^2(2a+x)=-x^3$.]

5. Some properties of the cissoid may be discovered by inversion of the parabola. In most of these that follow the drawing of the figure is left to the reader.

(*a*) Let the point C_1 on the cissoid be the inverse of the point C on the parabola. Let the tangent ZC to the parabola at C meet at D the tangent to the cissoid at C_1. Then, since the angle between two curves at a point of intersection is equal to the angle between the inverse curves at the inverse point, we have

$$\widehat{OCZ}=\widehat{DCC_1} \text{ (vertically opposite)}$$
$$=\widehat{DC_1C} \text{ (at the inverse point)}$$
$$\therefore\ DC=DC_1.$$

In particular OP bisects the angle between the tangents at P to the parabola and the cissoid.

(*b*) Let S be the focus of the parabola and X the foot of the directrix, that is, the point in which the directrix meets the axis of the parabola. Let the inverse of these points be S_1 and X_1 respectively. Through any point on the directrix two tangents can be drawn to the parabola and these tangents are

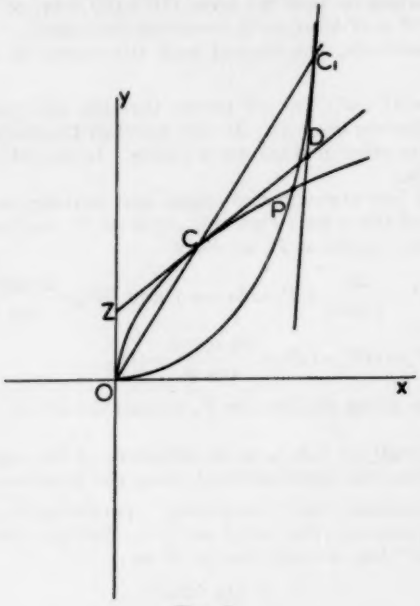


FIG. 2.

perpendicular. Hence, by inversion, through any point on the circle whose diameter is OX, two circles can be drawn to touch the cissoid and these two circles are orthogonal.

Further, the chord of contact of the tangents to the parabola passes through S. Hence the points of contact of the circles with the cissoid and the points O and S_1 are concyclic.

(*c*) The foot of the perpendicular from S to a tangent to the parabola at a point P lies on the tangent at the vertex of the parabola. Hence, the circle through O touching the cissoid at the point P_1 inverse to P and the circle which is orthogonal to it and passes through O and S_1 intersect again on Oy at the point inverse to the foot of the perpendicular.

(*d*) The circle through the three points of intersection of three tangents to a parabola passes also through the focus S.

Hence, three circles which pass through O and touch the cissoid intersect in three other points which lie on a circle passing through S_1.

(*e*) If LM is the latus rectum of the parabola, L_1 and M_1 the inverses of L and M respectively then, since the tangents at L and M to the parabola meet at right angles at X, it follows by inversion that the two circles which pass through O and touch the cissoid at L_1 and M_1 respectively are orthogonal and intersect again on the axis of the parabola at X_1.

In other words, a pair of orthogonal circles can be drawn through O and X_1 to touch the cissoid.

(*f*) Let the tangent at the vertex of the parabola meet XL and XM at Q and R respectively. Then the points X, Q, S and R are concyclic. By inversion QR cuts the pair of orthogonal circles of the previous section (*e*) at points Q_1 and R_1 respectively such that the points X_1, Q_1, R_1 and S_1 are concyclic.

(*g*) The original parabola and its conjugate are symmetrical about Oy. From a point A draw the tangents AB, AC to the original parabola and let these cut Oy at P and Q respectively. If D is the image of A in Oy then, of the tangents DE, DF drawn from D to the conjugate parabola one, say DE, will pass through P and the other, DF, will pass through Q.

The angles which the tangents at A subtend at the focus S are equal: similarly for the tangents from D to the conjugate. Also, by symmetry, each of the angles so subtended at one focus is equal to each of the angles at the other focus.

Let the tangents DE, DF to the conjugate meet the cissoid at G and H respectively. Then the angles OGD, OHD are right angles and the points G and H lie on the circle whose diameter is OD.

Hence we have

$$\begin{aligned}\widehat{GOH} &= \widehat{GDH}, \text{ in the same segment of the circle,}\\ &= \widehat{PAQ}, \text{ by symmetry,}\\ &= \widehat{BSA}, \text{ by the parabola property.}\end{aligned}$$

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MOTION IN A HORIZONTAL CIRCLE

The following error appears in a number of text-books, and tender minds are thereby led astray.

A body (such as a bicycle or railway carriage) is travelling in a horizontal circle with uniform speed; G is the centre of gravity of the body, O is the centre of the circle traced by G, GX is a horizontal line perpendicular to OG and GY is a vertical line.

The books in question state either that the sum of the moments about GX of the external forces vanishes, or that the resultant of these forces passes through G (which implies the vanishing of the sum of these moments). This sum, however, is actually equal to the square of the angular velocity of the body multiplied by its product of inertia about the axes GO and GY; in general, therefore, it does not vanish.

One book (1) treats the problem as a statical one; others (2, 3) give as the reason for the vanishing of the sum of the moments the absence of rotation about GX; in another case (4) no reason is given.

Bradford Technical College. J. G. Freeman

REFERENCES

(1) Geary, Lowry and Hayden: *Advanced Mathematics for Students of Technology*, Part I, 1945, p. 378.

(2) Humphrey and Topping: *A Shorter Intermediate Mechanics*, 1949, pp. 199, 200.

(3) Fawdry: *Dynamics*, 1943, p. 164.

(4) Ibid., p. 165.

A NET TO HOLD A SPHERE

By A. S. Besicovitch

Problem. A net is made of string round a unit sphere, so that the sphere cannot slip out of it. How long need the string be?

This problem was offered by me as a "Contest Problem" to undergraduates of Cambridge last Lent Term and a correct solution was given by Mr. R. Schwarzenberger of Trinity College. I am publishing my own solution, as his solution is considerably more complicated.

We may obviously assume all the knots to be on the surface of the sphere and all the links to be arcs of great circles. A precise description of such a net can be given in the following way. Divide the whole surface of the sphere into a finite number of distinct domains, each of which is a convex spherical polygon, bounded by a finite number of arcs of great circles with interior angles at the vertices less than π. Replacing every arc by a piece of string running along it, and making a knot wherever two or more strings meet, we shall get a net. The boundary of a domain will be called a *mesh* and the piece of string between a pair of consecutive knots a *link*.

Observe that a convex polygon is always contained in a hemisphere. If a net has a mesh consisting of two links only then the links are semi-circles and the sphere can slip out of such a net. Any other mesh is of length less than 2π.

Given a convex spherical polygon take a great circle containing one of its sides. Then apart from the side belonging to the circle the boundary of the polygon has no other points in common with the circle, and the whole of the polygon is included in one of the hemispheres into which the sphere is divided by the circle. Suppose now that two non-overlapping convex polygons have one side or a part of one side in common. Then they are included in two different hemispheres into which the sphere is divided by the great circle containing the common side (or part of a side) of the polygons and thus the boundaries of the polygons have no other points in common. Hence, *two non-overlapping convex polygons cannot have more than one side or a part of one side in common.*

Lemma. 1. *A closed curve Γ of length less than 2π on the unit sphere is separated from the centre of the sphere by a plane.*

Let A, B be a pair of points bisecting Γ, C the mid-point of an arc of the great circle joining A and B, and γ the equator of C. If D is any point of γ then the sum of the arcs of great circles $AD+BD$ is equal to π, and consequently such a point D cannot belong to either of the two arcs into which Γ is divided by A and B, since these arcs are of length less than π. Thus Γ does not meet the equator and any plane between Γ and the equator separates Γ from the centre.

Corollary 1. The unit-sphere cannot slip through a mesh of length less than 2π.

Corollary 2. If a net has no mesh of two links then the sphere cannot slip out of it.

A net that lets out the sphere if any one of its links is removed is called a minimum net.

Lemma 2. *If m_1 and m_2 are two meshes of length l_1 and l_2 of a minimum net having one link of length l in common (call the link also l) then $l_1+l_2-2l\geqslant 2\pi$.*

Observe that if we remove one link from a net, the remainder need not be a net in the sense of our definition. We shall call it a reduced net. If a reduced net is not contained in a hemisphere and if every mesh is less than 2π then the sphere cannot slip out of it. If $l_1+l_2-2l<2\pi$, so that the loop obtained from m_1 and m_2 by removing the common link is of length less than 2π, then it is contained in a hemisphere and so is the link l. On the other hand, the rest of

the net cannot be contained in the same hemisphere. The sphere cannot therefore slip out from the reduced net, contrary to the definition of a minimum net. Thus the lemma is proved.

Take now a mesh m_0. Let $l_1, l_2, \ldots, l_k$ be the links of m_0, and $m_1, m_2, \ldots, m_k$ the adjacent meshes. Denote by $\Sigma_{m_i} l$ the sum of links of m_i different from l_i.

By Lemma 2

$$l_1 + \ldots + l_k + \Sigma_{m_i} l \geqslant 2\pi + l_i.$$

Hence

$$(k-1)(l_1 + \ldots + l_k) + \sum_{i=1}^{k} \Sigma_{m_i} l \geqslant 2k\pi. \tag{1}$$

In the sum $\sum_{i=1}^{k} \Sigma_{m_i} l$ of the adjacent meshes every link appears once or twice.

Thus (1) can be written in the form

$$(k-1)(l_1 + l_2 + \ldots + l_k) + \Sigma a_j l_j' \geqslant 2k\pi,$$

where l_j' are the links of $m_1, m_2, \ldots m_k$ other than $l_1, l_2, \ldots l_k$ and $a_j = 1$ or 2.

Observing that $l_1 + \ldots + l_k < 2\pi$, we have

$$2(l_1 + \ldots + l_k) + 2\Sigma l_j' \geqslant 2k\pi - (k-3)(l_1 + \ldots + l_k).$$

Hence, for $k > 3$,

$$l_1 + \ldots + l_k + \Sigma l_j' > 3\pi,$$

and for $k = 3$

$$l_1 + \ldots + l_k + \Sigma l_j' \geqslant 3\pi,$$

where the sign of equality can appear only if every $a_j = 2$, and the net itself can be of length 3π, only if $l_1 + \ldots + l_k + \Sigma l_j'$ are all the links of the net, that is if m_0 is a triangle and m_0 together with the adjacent meshes m_1, m_2, m_3 form the whole of the net.

Then the boundaries of m_1, m_2, m_3 are three arcs of great circles through the vertices of m_0 concurrent at a point A outside m_0. From the condition that the net is not contained in a hemisphere, it follows that B, the antipode of A, lies inside m_0. Let $C_1C_2C_3$ be m_0. Take 3 semicircles through A, B and C_1, C_2, C_3 respectively.

We have

$$BC_1A + BC_2A + BC_3A = 3\pi,$$

and from $C_3C_2 + C_2C_1 > BC_2 + BC_1$ and the corresponding inequalities obtained by a cyclic change of suffixes,

$$BC_1 + BC_2 + BC_3 < C_2C_3 + C_3C_1 + C_1C_2.$$

Hence

$$AC_1 + AC_2 + AC_3 + C_2C_3 + C_3C_1 + C_1C_2 > 3\pi.$$

Thus a minimum net is always of length $> 3\pi$. By taking the triangle $C_1C_2C_3$ sufficiently small, we can make the length of the net as near 3π as we like.

Trinity College Cambridge A. S. B.

1882. The charge for water for a house of a rateable value £50 is 10·98825% of the rateable value.—On the back of a bill from the —— —— Valley Water Co. [Per Mr. A. W. Siddons.]

SOME CALCULATIONS IN LOGIC

By C. C. H. Barker

1. *Introduction.* The traditional two-valued logical calculus is concerned with *propositions* and their *truth-values, true* or *false*. Since the work of George Boole (1) these truth-values have been associated with the symbols 1, 0 respectively. If p denotes a proposition we use the same symbol p to denote its truth-value ; thus if p is true we write $p=1$ and if p is false we write $p=0$. Then derived propositions such as "not p", "p and q" have themselves truth-values which depend on those of p, q and are thus functions of the truth-values p, q.

The algebra of logic is essentially the algebra of classes and, since this is an unwieldy calculus in many respects, problems in elementary logic are usually solved by the use of "truth-tables"; these exhibit the value of a derived proposition for all possible combinations of values of the constituent propositions. However it is possible to set up a calculus of propositions closely allied to ordinary algebra. M. H. Stone (2) has defined a *Boolean ring* as a ring in which every element is idempotent, satisfying $xx=x$. He shows that such a ring is necessarily commutative and has the law $x+x=0$, and that if the ring has a unit e, the introduction of a binary operation U and a unary operation N through the equations

$$pUq=pq+p+q, \quad Np=p+e$$

converts the ring into an algebra of classes in which U denotes union and N negation.

The purpose of this note is to illustrate a method for dealing with problems in elementary logic based on the above relations. The method is then extended, in a direction quite remote from Boolean rings, to deal with three-valued logic. The results obtained are of the simplest type and are all known.

2. *The two-valued logic.* If we restrict a Boolean ring to two elements these are necessarily 0 and 1. The ring is then isomorphic to the field of the integral residues modulo 2, which has the laws (i) $xx=x$, (ii) $x+x=0$, and multiplication and addition tables as follows :

$$0\,.\,0=0, \quad 0\,.\,1=1\,.\,0=0, \quad 1\,.\,1=1\,;$$
$$0+0=0, \quad 0+1=1+0=1, \quad 1+1=0.$$

It may be noted that these tables can be interpreted as the laws of combination of odd and even integers, represented by 0 and 1 respectively.

We now suppose that the symbol p representing a proposition or its truth-value is an element of this field. The negative of p, denoted by Np, is false when p is true and true when p is false. Therefore

$$Np=p+1. \quad \text{.....(1)}$$

The compound proposition "p implies q", denoted by pCq, is false if p is true and q is false, and true otherwise. Let

$$pCq=\alpha pq+\beta p+\gamma q+\delta,$$

since this is the most general polynomial in p and q.

When $p=0$ and $q=0$, $pCq=1$; therefore $\delta=1$.
" $p=0$ " $q=1$, $pCq=1$; " $\gamma+\delta=1$.
" $p=1$ " $q=0$, $pCq=0$; " $\beta+\delta=0$.
" $p=1$ " $q=1$, $pCq=1$; " $\alpha+\beta+\gamma+\delta=1$.

Hence $\alpha=1$, $\beta=1$, $\gamma=0$, $\delta=1$.

Thus $$pCq=pq+p+1. \quad \text{.....(2)}$$

Clearly the general polynomial in m variables has 2^m terms and there are correspondingly 2^{2^m} truth-functions of m variables (including the two constants 0, 1).

Three other functions of p and q are defined as follows:

"p or q", denoted by pOq, which is true if at least one of p, q is true, and false if both are false;

"p and q", denoted by pAq, which is true if both p and q are true, and false otherwise;

"p is equivalent to q", denoted by pEq, which is true if p and q are both true or both false, and false otherwise.

By the same method we can represent these functions by the polynomials:

$$pOq=pq+p+q \quad \text{.....(3)}$$
$$pAq=pq \quad \text{.....(4)}$$
$$pEq=p+q+1 \quad \text{.....(5)}$$

In the formal development of the propositional calculus, these compound propositions are usually defined in terms of the unary symbol N and the binary symbol C, taken as primitive. The definitions are:

$$pOq=(pCq)Cq \quad \text{.....(3')}$$
$$pAq=N\{(Np)O(Nq)\} \quad \text{.....(4')}$$
$$pEq=(pCq)A(qCp) \quad \text{.....(5')}$$

As an example of the method we show that these definitions give (3), (4) and (5) respectively.

$$\begin{aligned}(pCq)Cq&=(pq+p+1)Cq, \quad \text{by (2)},\\ &=(pq+p+1)q+(pq+p+1)+1\\ &=pq+pq+q+pq+p+1+1\\ &=pq+p+q, \quad \text{which is (3).}\\ N\{(Np)O(Nq)\}&=(p+1)(q+1)+(p+1)+(q+1)+1, \quad \text{by (1) and (3),}\\ &=pq+p+q+1+p+1+q+1+1\\ &=pq, \quad \text{which is (4).}\\ (pCq)A(qCp)&=(pq+p+1)(qp+q+1)\\ &=p+q+1 \text{ on simplification, which is (5).}\end{aligned}$$

A *law* (or *tautology*) is a proposition which is always true, so that the corresponding polynomial is identically equal to 1. We give four examples of laws:

(The principle of deduction) $\{pA(pCq)\}Cq$ .....(6)

(Implication is transitive) $\{(pCq)A(qCr)\}C(pCr)$ .....(7)

(A false proposition implies any proposition) $(Np)C(pCq)$ .....(8)

(Reductio ad absurdum) $(pCq)E\{(Nq)C(Np)\}$ .....(9)

To prove (6) we have

$$\begin{aligned}pCq&=pq+p+1,\\ pA(pCq)&=p(pq+p+1)=pq,\\ \{pA(pCq)\}Cq&=(pq)q+(pq)+1=1.\end{aligned}$$

To prove (7) we have

$$\begin{aligned}pCq=pq+p+1, \quad qCr&=qr+q+1.\\ (pCq)A(qCr)&=(pq+p+1)(qr+q+1)\\ &=pq+qr+p+q+1.\\ \{(pCq)A(qCr)\}C(pCr)&=(pq+qr+p+q+1)(pr+p+1)\\ &\quad+(pq+qr+p+q+1)+1\\ &=1.\end{aligned}$$

The laws (8) and (9) may be proved similarly.

We can use this method to prove the symmetry of a logical expression. Thus $(pEq)Er = pE(qEr)$, since either side equals $p+q+r$, and either may be denoted simply by $pEqEr$. Similarly $pAqAr$ is unambiguous, and is equal to pqr.

We can now interpret an arbitrary function in logical symbols, for any function is a sum of products $p_1p_2\ldots p_k$, and we have

$$\begin{aligned} p_1p_2\ldots p_k &= p_1Ap_2A\ldots Ap_k, \\ p_1+p_2+\ldots+p_k &= p_1Ep_2E\ldots Ep_k \text{ if } k \text{ is odd} \\ &= N(p_1Ep_2E\ldots Ep_k) \text{ if } k \text{ is even.} \end{aligned}$$

Example 1. To find the logical expression which is true if just one of p, q, r is true and false otherwise.

The required expression x is given by

$$x = (pOqOr)A\{N(pAq)\}A\{N(qAr)\}A\{N(rAp)\}$$

We wish to simplify this, if possible. By substitution and simplification (or by determining the coefficients) we find

$$x = pqr + p + q + r.$$

Thus $x = pAqAr + pEqEr$, and the solution is

$$x = N\{(pAqAr)E(pEqEr)\}.$$

Example 2. To find the logical function x satisfying

$$xApAq = 0, \quad xO(pAq) = pOq.$$

Put $$x = \alpha pq + \beta p + \gamma q + \delta.$$

Then, substituting in the given conditions,

$$\begin{aligned} &(\alpha pq + \beta p + \gamma q + \delta)pq = 0, \\ &(\alpha pq + \beta p + \gamma q + \delta)pq + (\alpha pq + \beta p + \gamma q + \delta) + pq = pq + p + q. \end{aligned}$$

Simplifying gives

$$(\alpha+\beta+\gamma+\delta)pq = 0, \quad (\beta+\gamma+\delta)pq + (\beta-1)p + (\gamma-1)q = 0.$$

Both equations are satisfied if and only if

$$\alpha+\beta+\gamma+\delta = 0, \quad \beta+\gamma+\delta = 0, \quad \beta-1 = 0, \quad \gamma-1 = 0.$$

Thus $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 0$, giving $x = p+q$. The solution is therefore $x = N(pEq)$.

3. *A three-valued logic.* The following system for a logic with n truth-values is due to Lukasiewicz and is taken from Lewis and Langford (3). Propositions p have truth-values between 0 and 1, and the primitive functions are defined by

$$Np = 1 - p, \quad pCq = \min(1, 1 - p + q).$$

The functions pOq, pAq and pEq are defined by (3′), (4′) and (5′) as before, and so have numerical values as follows :

$$pOq = \max(p, q), \quad pAq = \min(p, q), \quad pEq = 1 - |p - q|.$$

For a three valued logic, where Lukasiewicz suggests 0, $\frac{1}{2}$, 1 we use 0, 1, 2 and define

$$\begin{aligned} Np &= 2 - p, \quad pCq = \min(2, 2 - p + q); \\ pOq &= \max(p, q), \quad pAq = \min(p, q), \quad pEq = 2 - |p - q|. \end{aligned}$$

We now suppose p to be an element of the field of integral residues modulo 3 which has the laws (i) $x^3 = x$, (ii) $x + x + x = 0$. The general polynomial in p has the form $\alpha p^2 + \beta p + \gamma$, and there are 27 of these (including 3 constants). Clearly now $Np = 2p + 2$. Several other functions of one variable, the " modal functions ", are defined, *e.g.* Mp (" p is possible ") defined by $Mp = (Np)Cp$ which then has the expression $Mp = 2p^2$.

The polynomials corresponding to the binary functions are :

$$\begin{aligned} pCq &= 2p^2q^2 + 2p^2q + 2pq^2 + pq + 2p + 2, \\ pOq &= p^2q^2 + p^2q + pq^2 + 2pq + p + q, \\ pAq &= 2p^2q^2 + 2p^2q + 2pq^2 + pq, \\ pEq &= p^2q^2 + p^2q + pq^2 + 2pq + 2p + 2q + 2. \end{aligned}$$

A law is now an expression which has identically the truth-value 2. We give one example, concerning the rule of inference in this logic.

Example 3. To show that $pC\{(pCq)Cq\}$ is a law, but $\{pA(pCq)\}Cq$ is not a law.

$pC\{(pCq)Cq\}$ is equal to $pC(pOq)$ *i.e.* $pC\max(p, q)$, which is a law from the definition. It may be verified that the corresponding polynomial has identically the value 2.

To evaluate $\{pA(pCq)\}Cq$ put $x = pCq$, $y = pAx$, $z = yCq = \{pA(pCq)\}Cq$. Then after substitution and simplification we find

$$\begin{aligned} x &= 2p^2q^2 + 2p^2q + 2pq^2 + pq + 2p + 2, \\ x^2 &= 2p^2q^2 + p^2 + pq^2 + 2p + 1, \\ y &= 2p^2q + 2p^2 + pq + 2p, \\ y^2 &= 2p^2q^2 + 2p^2 + pq^2 + 2p, \\ z &= 2p^2q^2 + p^2 + 2pq^2 + p + 2. \end{aligned}$$

Thus z is not identically equal to 2 and so z is not a law. In fact

$$\begin{aligned} z &= (p^2 + p)(2q^2 + 1) + 2 \\ &= 2p(p + 1)(q^2 + 2) + 2 \\ &= 2p(p + 1)(q + 1)(q + 2) + 2. \end{aligned}$$

Thus $z = 2$ if $p = 0$ or $p = 2$ or $q = 2$ or $q = 1$, while if $p = 1$ and $q = 0$ then $z = 1$.

4. *The general case.* The method of representation by polynomials can be extended to a logic with any finite number n of truth-values. The simplification resulting from working with integral residues can, however, only be obtained when these form a field, *i.e.* when n is prime. For example, in the four-valued logic analogous to that just considered we can represent pOq by

$$\begin{aligned} pOq = -\tfrac{1}{6}p^3q^3 + \tfrac{3}{4}(p^3q^2 + p^2q^3) - \tfrac{3}{4}(p^3q + pq^3) - \tfrac{7}{4}p^2q^2 \\ + \tfrac{15}{4}(p^2q + pq^2) - \tfrac{23}{6}pq + p + q. \end{aligned}$$

But the coefficients cannot be reduced to integers and therefore the subsequent calculations are laborious.

University College, Bangor. C.C.H.B.

REFERENCES

1. George Boole, *The Laws of Thought* (1854).

2. M. H. Stone, " The Theory of Representations for Boolean Algebras ", *Trans. American Math. Soc.*, 40 (1936), 37.

3. C. I. Lewis and C. H. Langford, *Symbolic Logic* (New York, 1932).

CORRESPONDENCE

TO THE EDITOR OF THE *Mathematical Gazette*

DEAR SIR,

Readers of the *Gazette* may be interested in a new technique for solving trigonometrical equations, invented by a candidate for an A level examination.

The problem was to show that, if a ladder rests with one end on the ground and the other end against a wall, and the friction is limiting at each end, then $\alpha = \frac{1}{2}\pi - 2\lambda$, where α is the inclination of the ladder and λ the angle of friction.

The candidate obtained the incorrect equation :

$$2 \tan \alpha = \cot \lambda - 3 \tan \lambda,$$

and then proceeded thus : taking tangents (he presumably meant inverse tangents) of both sides,

$$2\alpha = 180 - \lambda - 3\lambda,$$

so

$$\alpha = 90 - 2\lambda.$$

Yours etc,

E. J. F. PRIMROSE

TO THE EDITOR OF THE *Mathematical Gazette*

DEAR SIR,

In Note 2510 (*Mathematical Gazette*, May, 1955) Dr. T. A. S. Jackson asserts that two generally accepted laws of static friction are sometimes inconsistent. These laws are (1) that the reaction must lie within the cone of friction and (2) that friction opposes the tendency to relative motion. He supports his argument by considering the problem of a bar resting obliquely against a rough wall and rejects as erroneous a solution given by Loney. However, the forces assumed by Loney are in equilibrium and moreover the reaction at the wall satisfies both laws (1) and (2). I fail to detect any inconsistency. What Dr. Jackson's analysis does establish is that, if we are prepared to waive condition (2), then more oblique positions of the rod are possible than if we require that both conditions shall be satisfied. But, whether we are entitled to abandon the second law and solve all such problems under the condition imposed by the first law alone, as Dr. Jackson does, is a question which can only be decided by experiment. Dr. Jackson admits that the second law is accepted by most authorities and it would appear to be highly plausible. However, a factual statement from a physicist would be helpful in deciding the matter.

In problems where a rigid body is in contact with a rough plane at a number of points and is acted upon by a steadily increasing force R (see Note 2606, *Mathematical Gazette*, May, 1956), it may be objected that before limiting conditions are attained the direction of the relative motion tendency at a point of contact is indeterminate, so that law (2) cannot be applied. However, when conditions are limiting this is not the case and law (2) may be employed to determine these conditions and R uniquely. Since the body cannot be ideally rigid, it seems reasonable to suppose that when R has increased to this critical value, slipping will take place. This is not the point of view taken by Dr. Jackson in Note 2606, but only experiment can decide whether or not it is justified. No logical inconsistency appears to be involved.

Yours etc,

DEREK F. LAWDEN

MATHEMATICAL NOTES

2677. Particular Integrals of Partial Differential Equations.

The method for finding particular integrals explained in Note 2582 can be applied to linear partial differential equations with constant coefficients when the right hand side is of the forms

$$e^{ax+by}, \quad \sin(ax+by), \quad \cos(ax+by).$$

Consider, for example, the equation in the usual notation

$$F(D, D')z = e^{ax+by}.$$

A particular integral is easily obtained using operator method

$$z = e^{ax+by}/F(a,b), \qquad F(a,b) \neq 0.$$

When $F(a,b)=0$, the device explained in the above note gives a particular integral

$$z = \left[\frac{\partial}{\partial\alpha}\,(e^{ax+\beta y})\Big/\frac{\partial}{\partial\alpha}F(\alpha,\beta)\right]_{\substack{\alpha=a\\ \beta=b}}, \quad \left[\frac{\partial}{\partial\alpha}F(\alpha,\beta)\right]_{\substack{\alpha=a\\ \beta=b}} \neq 0,$$

and another could be obtained by differentiation with respect to β. The verification that these are particular integrals follows the same pattern as that for ordinary differential equations.

The method can be used to obtain a particular integral of a homogeneous equation of the form

$$(D - mD')^n z = \psi(y + mx)$$

in the following way; it is well known (see "Differential Equations" by Piaggio) that

$$\frac{1}{F(D, D')}\,\phi^{(n)}(ax+by) = \frac{1}{F(a, b)}\,\phi(ax+by), \quad F(a, b) \neq 0$$

so that

$$z = \frac{1}{(\alpha - m)^n} f(y + \alpha x), \quad \alpha \neq m$$

where

$$f^{(n)}(y + \alpha x) = \psi(y + \alpha x) \tag{1}$$

is a solution of

$$(D - mD')^n z = \psi(y + \alpha x).$$

By the same tentative approach as that explained in Note 2582 we are led to infer that

$$z = \left[\frac{\partial^n}{\partial\alpha^n} f(y+\alpha x)\Big/\frac{\partial^n}{\partial\alpha^n}(\alpha - m)^n\right]_{\alpha=m} \qquad \text{.....(2)}$$

is a solution of

$$(D - mD')\, z = \psi(y + mx)$$

and the verification that this is so is carried out in exactly the same manner. Further we have

from (2)

$$z = \frac{x^n f^{(n)}(y+mx)}{n!}$$

and using (1)

$$z = \frac{x^n}{n!}\psi(y+mx)$$

which is the required form of the particular integral.

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2678. Identities in sums of powers of integers

1. The familiar identity

$$\Sigma n^3 = (\Sigma n)^2$$

may be made intuitively obvious in the following way.

Let $T_n = \sum_1^n r = \frac{1}{2}n(n+1)$. Divide the T_n odd integers from 1 to $2T_n - 1$ into groups of 1, 2, 3, ... , n terms as follows:

$$1 + (3+5) + (7+9+11) + \ldots + \{(2T_{n-1}+1) + (2T_{n-1}+3) + \ldots + (2T_n - 1)\}.$$

The sum of the terms in the rth group is

$$\tfrac{1}{2}r\{r(r-1) + 1 + r(r+1) - 1\} = r^3$$

and the sum of all the terms is $\frac{1}{2}T_n \cdot 2T_n = T_n^2$ as required.

2. If we divide the odd numbers from 1 to $4T_n - 1$ into groups of 2, 4, 6, ... , $2n$ terms, thus

$$(1+3) + (5+7+9+11) + (13+15+17+19+21+23) + \ldots + (4T_{n-1}+1 + \ldots + 4T_n - 1)$$

we see that the sum of the terms in the rth group is

$$r \cdot 4(T_r + T_{r-1}) = 4r^3$$

and therefore

$$1 + 3 + 5 + \ldots + (4T_n - 1) = 4(1^3 + 2^3 + \ldots + n^3),$$

which shows that the sum of the first $n(n+1)$ odd integers is four times the sum of the first n cubes, and provides another proof of (1).

3. Divide the integers from 1 to $2n^2 - 1$ into groups of 1, 6, 10, 14, ... , $4n - 2$ terms :

$$1 + (2+3+4+5+6+7) + (8+9+10+\ldots+17) + \ldots + \{2(n-1)^2 + \ldots + 2n^2 - 1\}.$$

The sum of the terms in the rth group is

$$(2r-1)(2r^2 - 4r + 2 + 2r^2 - 1) = (2r-1)^3$$

and so
$$1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = 1 + 2 + 3 + \ldots + (2n^2 - 1).$$

4.
$$6\{1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2\} + 2n = (2n)^2,$$

$$6\{2^2 + 4^2 + 6^2 + \ldots + (2n)^2\} + 2n + 1 = (2n+1)^2 ;$$

for
$$(2n+2)^3 - (2n)^3 = 6(2n+1)^2 + 2$$

and
$$(2n+3)^3 - (2n+1)^3 = 6(2n+2) + 2.$$

5. If $V_n = \sum_{r=1}^{n-1} T_r = \frac{1}{6}(n-1)n(n+1)$ then

$$V_{2n} - 2V_n = n^3 ;$$

for
$$V_{2n+2} - V_{2n} = T_{2n} + T_{2n+1} = (2n+1)^2,$$

$$2(V_{n+1} - V_n) = 2T_n = n(n+1)$$

and
$$(2n+1)^2 - n(n+1) = (n+1)^3 - n^3.$$

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2679. A test for prime numbers

Let $S_n^k(x)$ denote the sum obtained by replacing each of the first k terms in the series

$$1+3+5+\ldots+(2n-1)$$

by the number x, so that

$$S_n^k(x)=x+x+\ldots+x+(2k+1)+(2k+3)+\ldots+(2n-1)=kx+(n^2-k^2),\ k<n.$$

Then

THEOREM 1. The number $8N-1$ is composite if and only if there are numbers k, n with $k<n$ such that

$$\tfrac{1}{2}N=S_n^k(\tfrac{1}{2}) \quad \text{or} \quad \tfrac{1}{2}N=S_n^k(-\tfrac{1}{2}).$$

PROOF. If $\frac{1}{2}N=S_n^k(\frac{1}{2})$ then

$$8N-1=16(n^2-k^2)+8k-1=\{4(n-k)+1\}\ \{4(n+k)-1\}$$

and if $\frac{1}{2}N=S_n^k(-\frac{1}{2})$ then

$$8N-1=16(n^2-k^2)-(8k+1)=\{4(n+k)+1\}\ \{4(n-k)-1\},$$

so that $8N-1$ is composite. Conversely, if $8N-1$ is composite, it must have a factor of the form $4p-1$; if the second factor is $4q+1$ then $p-q$ is divisible by 2 and so $p-q=2k$ for a certain k.

Taking $n=p-q$ we have

$$8N-1=\{4(n+k)-1\}\{4(n-k)+1\}$$

so that $\frac{1}{2}N=\frac{1}{2}k+(n^2-k^2)=S_n^k(\frac{1}{2})$ or $S_n^{-k}(-\frac{1}{2})$ according as k is positive or negative.

Companion results to Theorem 1 hold for numbers of the forms $8N+1$, $8N\pm3$. Let

$$\Sigma_n^k(x)=x+x+\ldots+x+(k+1)+(k+2)+\ldots+n=kx+\tfrac{1}{2}(n-k)(n+k+1)$$

then

THEOREM 2. The number $8N+3$ is composite if and only if there are numbers k, n such that $\frac{1}{4}N=\Sigma_n^k(\frac{1}{4})$, $k<n$, or $\frac{1}{4}N=\Sigma_n^k(\frac{3}{4})$, $k\leqslant n$.

PROOF. If $\frac{1}{4}N=\Sigma_n^k(\frac{1}{4})$ then $8N+3=8k+3+16(n-k)(n+k+1)$
$=\{4(n-k)+1\}\ \{4(n+k)+3\},$

and if $\frac{1}{4}N=\Sigma_n^k(\frac{3}{4})$ then $8N+3=24k+3+16(n-k)(n+k+1)$
$=\{4(n+k)+1\}\ \{4(n-k)+3\}.$

Conversely if $8N+3$ is composite then one factor must be of the form $4p+1$ and the other $4q+3$ so that $p+q$ is even; if $p+q=2k$ and $n=p-k$ then

$$8N+3=\{4(n+k)+1\}\ \{4(n-k)+3\}$$

so that $\frac{1}{4}N=\Sigma_n^k(\frac{3}{4})$ or $\Sigma_n^{-k}(\frac{1}{4})$ according as k is positive or negative. (We note that if $q=0$ then $k=n$).

The results for numbers of the forms $8N+1$, $8N-3$ are as follows. If $P_n^k(x)$, $Q_n^k(x)$ are respectively the sums obtained by replacing each of the first k terms in the series

$$1+5+\ 9+\ldots+(4n-3),$$
$$3+7+11+\ldots+(4n-1),$$

by the number x, so that

$$P_n^k(x) = kx + (n-k)(2n+2k-1) \quad \text{and} \quad Q_n^k(x) = kx + (n-k)(2n+2k+1)$$

then

THEOREM 3. The number $8N+1$ is composite if and only if there are numbers k, n with $k<n$ such that either $N = P_n^k(-1)$ or $N = Q_n^k(+1)$, $8N+1$ having the factors $\{4(n-k)-1\}\{4(n+k)-1\}$ in the first case and $\{4(n-k)+1\}\{4(n+k)+1\}$ in the second.

THEOREM 4. The number $8N-3$ is composite if and only if there are numbers k, n with $k<n$ such that either $N = P_n^k(+1)$ or $N = Q_n^k(-1)$, $8N-3$ having the factors $\{4(n-k)+1\}\{4(n+k)-3\}$ in the first case and $\{4(n-k)-1\}\{4(n+k)+3\}$ in the second.

The proofs of Theorems 3 and 4 follow closely the proof of Theorem 1 and are omitted.

The author thanks the Editor for his help in preparing this note for publication and for providing proofs of the results it contains.

A. N. NICHOLSON

2680. Problems on "variable mass" : a scholarship teaching note.

Newton when he had, as he said, "the whole ocean of truth lying uncharted before him" succeeded in mapping extremely well when he stated his second law of motion in terms of rate of change of momentum. We find for example the electron fitting neatly into this formulation, and we can set our science pupils to integrate the uniform field equation :

$$\frac{d}{dt}\left\{\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right\} = k$$

The success of the rate-of-change-of-momentum routine has led unfortunately to a number of unsound applications, which are often classed together as "variable mass problems". Two examples will be given, for both of which an unsound treatment has been invited by the wording of questions at S-level by one of our examining boards.

EXAMPLE 1. A falling drop of water is gaining mass by aggregation from the surrounding medium according to a given law. Given the initial mass m_0, to find an expression for the velocity at time t.

The unsound short cut is to consider the drop a mechanical system and to equate its instantaneous weight to the rate of change of "its" downward momentum. This gives the right result *provided the mass gained by accretion was previously at rest.*

A correct treatment must deal with a system of particles clearly defined, into which no new particles come during the discussion. For beginners, as well as for others who like a clear picture of what is going on, it seems advisable to consider a short interval between time t and $t+\delta t$. The system consists of the water which makes up the complete drop at the later instant, the mass being $m+\delta m$, of which δm was picked up during the interval. The drop's velocity is v at time t and $v+\delta v$ at time $t+\delta t$. To emphasise the importance of correct principle the swept-up mass is supposed to be falling with velocity u.

We equate the increase in downward momentum of the system to the impulse of external force on it (and there is no awkwardness about force between m and δm because such forces are internal). We write:

$$(m+\delta m)(v+\delta v) - (\delta m \,.\, u + mv) = mg\,\delta t$$

and the differential equation follows. For solution we must know the law of aggregation, e.g. : $m = m_0 + kt$.

EXAMPLE 2. A rocket is rising vertically against negligible air resistance. Mass issues at constant rate k, at constant *relative* velocity U.

A correct treatment considers the rocket and its contents at time t and total mass m as the " system ". After lapse of time this is in two parts,

(i) mass $(m - k\delta t)$ with upward velocity $v + \delta v$, and
(ii) mass $k\delta t$ with upward velocity $v - U$.

Equating the increase in upward momentum of the system to the impulse of the upward force, which need only be stated to the first order, we have

$$(m - k\delta t)(v + \delta v) + k\delta t(v - U) - mv = -mg\delta t$$

which gives

$$m\frac{dv}{dt} = kU - mg$$

where $m = m_0 - kt$.

The constant term kU is the rocket-propulsive force of the rocket engineer. In a recent examination paper this was introduced as a given constant force F. The boy rocket-enthusiast who then wrote down $F - mg$ equal to $m\frac{dv}{dt}$ in this instance disagreed with examiner, whose printed result required the automatic use of $\frac{d}{dt}(mv)$. As in example 1, there are special conditions in which this is correct ; in this case, that the gases after efflux at every stage of the motion have no momentum. It is clear therefore that if a correct equation is to be derived the student must be told the whole story and must then equate impulse to change of momentum for a correctly constituted system.

In conclusion, after pointing out some pitfalls in the paths of those who frame as well as of those who try to solve such problems one would like to challenge those who have read so far, to frame a question about an express train picking up water at speed (a) when the driver takes action to maintain his speed steady, and (b) when he does not. To state the assumptions so as to make the question soluble is no mean intellectual exercise.

St. Paul's School. A. J. MOAKES

2681. The representation of projectivities.

1. If we are given a family F of geometrical objects, it may be possible to set up a (1, 1) correspondence between the members of F and the members of another family F' of geometrical objects. If so, the members of F are said to be *represented* by members of F'. Only in certain cases does the representation lead to results of geometrical interest.

For example, in Vol. XXI of the *Gazette*, E. A. Maxwell (pp. 46–49) and D. Pedoe (pp. 210–215) showed how to represent the family of circles in a plane by the family of points of a 3-dimensional Euclidean space. The results show many interesting connections between the geometry of circles in a plane and the geometry of 3-dimensional space.

In this article, we consider the representation of the family of projectivities of a 1-dimensional projective space by the points of a 3-dimensional space.

2. The equation

$$x\lambda\mu + y\lambda + z\mu + t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

defines a correspondence between two non-homogeneous parameters λ and μ. If $xt - yz \neq 0$, the correspondence is (1, 1) and is said to be non-singular : if $xt - yz = 0$, it is said to be singular.

Let L be a variable element of a projective space l of 1 dimension (for instance, l could be a line, and L one of its points), and let L have parameter λ, referred to a suitable reference system on l. Similarly, let M be a variable element of a projective space m of 1 dimension, and let M have parameter μ. Then (1) defines a correspondence between L and M, called a projectivity π. We shall write $M = \pi(L)$.

Two cases can arise : l and m may be distinct or coincident. We show how to represent projectivities by points of a 3-dimensional projective space, in such a way that the two cases may be treated simultaneously.

Much of this work must be familiar to geometers, but I can find no account of it.

3. We represent π by the point P, with homogeneous coordinates (x, y, z, t). Then singular projectivities are represented by points of the quadric S, whose equation is

$$xt - yz = 0.$$

We shall assume henceforth that all projectivities mentioned are non-singular.

We give first a geometrical interpretation of the representation, in which l and m are distinct.

Let l and m be the two systems of generators of S, so that L is the generator whose equations are

$$\lambda x + z = 0, \quad \lambda y + t = 0,$$

and M is the generator whose equations are

$$\mu x + y = 0, \quad \mu z + t = 0.$$

We shall call these a λ-generator and a μ-generator respectively. Then, if λ and μ are related by (1), the plane containing these generators is given by (1), and therefore *the plane containing L and M passes through P.*

This result gives us a procedure for finding $\pi(L)$ if π and L are given. It is the μ-generator in which the plane containing P and L cuts S again.

4. In this section, we give a geometrical interpretation of the representation, in which l and m are coincident. We first note that

(a) the identity is represented by the point $(0, 1, -1, 0)$, which we call O,

(b) an involution is characterised by the relation $y - z = 0$, and so involutions are represented by points of the polar plane ω of O with respect to S.

Let l and m both be the conic ϕ in which ω meets S, and let L and M have coordinates $(1, -\lambda, -\lambda, \lambda^2)$ and $(1, -\mu, -\mu, \mu^2)$ respectively. Then, if λ and μ are related by (1). *the plane containing the λ-generator through L and the μ-generator through M passes through P.*

This result gives us a procedure for finding $\pi(L)$ if π and L are given. It is the point where the plane containing P and the λ-generator through L meets ϕ again. (Since l and m are coincident, we can consider the reverse correspondence π^{-1}. It will be seen that $\pi^{-1}(M)$ is the point where the μ-generator through M meets ϕ again.)

We now prove some further results of this interpretation.

(i) A given point L of ϕ is a double point of π if and only if

$$x\lambda^2 + (y + z)\lambda + t = 0,$$

which means that P lies in the tangent plane to S at L.

(ii) If π is not an involution, then, since PLM is a tangent plane of S, the envelope of LM is the conic ϕ' cut on ω by the tangent cone from P to S. It follows that ϕ' touches ϕ at the points where ϕ is met by the polar plane of P with respect to S. The result of (i) shows that these are the double points of ϕ.

If π is an involution, P, L and M lie in ω and in a tangent plane of S, so LM passes through P: in this case, ϕ' degenerates (as an envelope) into the point P counted twice. Again, the double points of π are the points where ϕ is met by the polar plane of P with respect to S (in this case, more simply, the points where ϕ is met by the polar of P with respect to ϕ).

(iii) π has a single double point if and only if

$$(y+z)^2 - 4xt = 0.$$

Such projectivities are therefore represented by points of Ω, the tangent cone from O to S. It follows that, if L is the single double point of such a projectivity, then P lies on OL.

5. Some standard theorems about projectivities in a 1-dimensional space can now be interpreted in the 3-dimensional space.

(i) If π and π' are projectivities of l, there are, in general, two pairs of elements of l which correspond in π and in π'. This is equivalent to the result that, in general, two tangent planes to a quadric S pass through two given points, P and P'. In fact, the latter result could provide a simple proof of the theorem.

(ii) π and π' are commutative (that is, the projectivities obtained by combining π and π' in either order are identical) if and only if *either* π and π' have the same double elements, or π and π' are involutions, whose double elements are harmonically conjugate. Now if π and π' have the same double elements, it follows from (ii) of section 4 that the polar line of PP' with respect to S lies in ω, and so PP' passes through O. If π and π' are involutions, whose double elements are harmonically conjugate, then P and P' lie in ω and are conjugate with respect to ϕ.

6. We could, of course, represent π by a plane, rather than a point, of a 3-dimensional space. This representation would be simply the dual of the one which we have given.

7. As Dr. D. B. Scott remarked, in a letter to me, the representation may be extended to higher correspondences. It may be shown, for example, that if L and M are interpreted as in section 3, and λ and μ are related by a given (2, 2) correspondence, then the plane containing L and M touches a quadric Σ. Since the plane also touches S, it touches every quadric of the tangential pencil determined by S and Σ.

E. J. F. PRIMROSE

2682. Interchanged poles and polars.

J. Leech, in Note 2397, *Gazette*, May 1954, p. 118, has pointed out a curious relation between poles and polars in a triangle ABC. With respect to the self-polar circle, the poles of the centroid G and the orthocentre H are the radical axis Δ of the circum-and nine-point circles and the line at infinity L; in the harmonic polarity with respect to the triangle ABC, the poles of G and H are L and Δ.

J. Leech asks if this interchange is significant.

In fact the considered relation does not characterize the elements involved, as it is a particular case of a general property: If, in a triangle ABC, α, β, γ are the barycentric coordinates of the centre M of a self-polar conic Σ and, if D and l are the polars, with respect to Σ, of the point P having as barycentric coordinates x, y, z and the point Q having as barycentric coordinates α/x, β/y, γ/z, then the polars of P and Q, with respect to the triangle ABC, are l and D.

When M is the Lemoine point, P and Q are counter-points.

St. Andre-lez-Bruges, Belgium. R. GOORMAGHTIGH

2683. On "nedians"

It is well known that the area of the triangle having as sides the medians of a triangle ABC equals three quarters of the area S of ABC.

J. Satterly (*Gazette*, May 1954, p. 111, May 1956, p. 109) has shown that, more generally, if D, E, F are points on BC, CA, AB respectively such that

$$BD/BC = CE/CA = AF/AB = 1/N,$$

the area of the triangle having as sides the "nedians" AD, BE, CF is

$$(N^2 - N + 1)S/N^2.$$

The expression has been obtained by Coordinate Geometry. The following proof is quite elementary.

We refer to figure 3 on page 110, *Gazette*, May 1956 (in that figure, the points E and F have to be interchanged). Then

$$\triangle ADH = \triangle ADC + \triangle ACH - \triangle DCH.$$

But

$$\triangle ADC = \frac{DC}{BC}S, \quad \triangle ACH = \frac{AF}{AB}S,$$

$$\triangle DCH = \frac{CH \,.\, DC}{AB \,.\, BC}S = \frac{AF \,.\, DC}{AB \,.\, BC}S,$$

this last relation being deducted from the fact that DCH and ABC have two of their sides parallel to each other.

Hence

$$\triangle ADH = \left(\frac{N-1}{N} + \frac{1}{N} - \frac{N-1}{N^2}\right)S = (N^2 - N + 1)S/N^2.$$

R. GOORMAGHTIGH

Editorial Note. Solutions have also been received from Mr. A. K. Rajagopal, Professor K. P. Pillai and Mr. J. E. Kingsland.

2684. On the Fibonacci sequence.

The object of this note is to show that the sum of the squares of consecutive Fibonacci numbers is itself a Fibonacci number, as is the difference of the squares of two Fibonacci numbers which are separated by a Fibonacci number. Let a_n denote the nth Fibonacci number so that $a_1 = a_2 = 1$ and

$$a_{n+1} = a_n + a_{n-1}$$

for all $n \geqslant 1$. A simple induction over l shows that

$$a_{n+l} = a_{l+1}a_n + a_l a_{n-1}. \qquad \text{.....(i)}$$

THEOREM 1. $$a_{2n} = a_{n+1}^2 - a_{n-1}^2$$

PROOF. Let $l = n$ in (i) then

$$\begin{aligned} a_{2n} &= a_{n+1}a_n + a_n a_{n-1} \\ &= a_n(a_{n+1} + a_{n-1}) \\ &= (a_{n+1} - a_{n-1})(a_{n+1} + a_{n-1}) \\ &= a_{n+1}^2 - a_{n-1}^2 \end{aligned}$$

THEOREM 2. $$a_{2n+1} = a_{n+1}^2 + a_n^2$$

PROOF. Let $l = n + 1$ in (i) then

$$\begin{aligned} a_{2n+1} &= a_{n+2}a_n + a_{n+1}a_{n-1} \\ &= (a_{n+1} + a_n)a_n + a_{n+1}(a_{n+1} - a_n) \\ &= a_{n+1}a_n + a_n^2 + a_{n+1}^2 - a_{n+1}a_n \\ &= a_{n+1}^2 + a_n^2 \end{aligned}$$

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2685. A simple device for testing primality.

The elementary method for testing the primality of a number N, is to divide it by all primes $\leqslant\sqrt{N}$, and by considering the choice of final digits the number of primes to be so tested can be restricted. This method, possibly with some refinements, is used by the electronic computing machine for testing large numbers.

If P_L is the last prime in the Lehmer Table of Primes, then all numbers not greater than P_L^2 can be tested. But how are we to test a number such as $N_0 = 10P_L^2 + D$, where D is an odd digit, not 5? The machine could, of course, construct all primes $\leqslant\sqrt{10P_L^2 + D}$ in order to test N_0, but even with the machine this procedure is laborious and costly.

We shall state a most simple device, which may be of general and pedagogic interest, but which could also be used by the *computing machine* with a considerable saving of labour.

Lemma. The number $10c + d$ divides $10C + D$ ($D, d, = 1, 3, 7, 9$) if it divides $C - T(c)$, where $T(c)$ is a linear in c depending only on d and D.

The 16 values of $T(c)$ are arrayed in the Table below

		VALUE OF D			
		1	3	7	9
VALUE OF d	1	c	$3c$	$7c$	$9c$
	3	$7c+2$	c	$9c+2$	$3c$
	7	$3c+2$	$9c+6$	c	$7c+4$
	9	$9c+8$	$7c+6$	$3c+2$	c

It will be seen that if $N = 10C + D = N_1N_2 = (10c_1 + d_1)(10c_2 + d_2)$

$$\text{then } C - T(c_1) = c_2N_1$$

$$\text{and } C - T(c_2) = c_1N_2$$

For example, let us test 72433. We need only test for primes $10c_1 + 3$ and $10c_2 + 7$. With $d = 3$, $D = 3$ we have $T(c) = c$ so that when we reach the prime 113 for which $c = 11$, we have

$$7243 - 11 = 113 \times 64$$

and so $c_2 = 64$ and $10c_2 + 1 = 641$ and thus

$$72433 = 113 \times 641.$$

It is equally obvious that if $C - T(c)$ is prime for all c then N is prime.

The question how can we test for the number $N_0 = 10P_L^2 + D$ is now easily answered. M. RUMNEY

2686. Tests of Divisibility.

In Gazette Vol. XXXIX, No. 329, September 1955, a new test of divisibility by 19 was given. Since then more tests for divisibility by 7, 13 and 17 have been found.

The number N to be tested is expressed in the form $10a+b$. Then

N is divisible by 7 if $a-2b$ is divisible by 7;
N „ „ „ 13 „ $a+4b$ „ „ „ 13;
N „ „ „ 17 „ $a-5b$ „ „ „ 17.

These tests are contained in the following general statements:
If the divisor M is of the forms $10p+1$, $10p+3$, $10p+7$, $10p+9$, respectively, then N is divisible by M if $a-pb$, $a+(3p+1)b$, $a-(3p+2)b$, $a+(p+1)b$, respectively are divisible by M, e.g. N is divisible by $53 (M=53=10\times 5+3)$ if $a+16b$ is divisible by 53 $(16=5\times 3+1)$.

The proofs of these are based on simple congruences.

Kamachumu, Bukoba, East Africa. J. A. K. KASHANGAKI

2687. On $\Sigma r^3=(\Sigma r)^2$.

Most students are, I think, somewhat surprised when $\sum_1^n r^3$ turns out to be $\frac{1}{4}n^2(n+1)^2=(\sum_1^n r)^2$. The following simple, direct demonstration may add interest to this result. It can, of course, be used as an alternative method of *proving* the formula for Σr^3, if so desired.

Let $S_2(r)\,(r\geqslant 2)$ be the sum of all the possible products formed two at a time from the first r integers.

[Thus $S_2(4)=1.2+1.3+2.3+1.4+2.4+3.4$.]

Clearly, (if $r\geqslant 3$),

$$S_2(r)-S_2(r-1)=r\sum_1^{r-1} k$$
$$=\tfrac{1}{2}(r^3-r^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Define $S_2(0)=S_2(1)=0$; then (1) holds for $r\geqslant 1$.

Summing (1), then from $r=1$ to n,

$$2S_2(n)=\sum_1^n r^3-\sum_1^n r^2.$$

But obviously,
$$2S_2(n)=(\sum_1^n r)^2-\sum_1^n r^2.$$

Hence the result. ROGER F. WHEELER

2688. Relations between coaxal circles and (i) a hyperbola, (ii) an ellipse.

The following simple results arise naturally from the equations of the hyperbola and the ellipse.

Let $X'OX$, $Y'OY$ be perpendicular Cartesian axes and on $X'OX$ take $A'O=OA=a$; $B_1'O=OB_1=b$. On $Y'OY$ take $B'O=OB=b$ and let $\tan\alpha=a/b$. Further, let any circle through B_1', B_1 cut $Y'OY$ at Q', Q and

(*a*) draw the isosceles triangle QPQ' with $PQQ'=PQ'Q=\alpha$;
(*b*) let $A'Q'$, QA meet in R; $A'Q$, AQ' in R';
(*c*) let Q_1 be the reflection of Q' in $X'OX$ and let $A'Q_1$, QA meet in S; AQ_1, $A'Q$ in S'.

Then the locus of P, R, R' is the hyperbola

$$x^2/a^2-y^2/b^2=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and that of S, S' is the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \quad \ldots\ldots(2)$$

(*d*) If R, any point on (1), be joined to A, A', the vertices of (1), RA, RA' cutting the conjugate axis $Y'OY$ in Q, Q', then Q, B_1, Q', B_1' all lie on a circle Σ.

(*e*) The meets of perpendiculars from (i) Q to RA' (ii) Q' to RA lie on Σ.

Proof. If $OQ = \mu b$ (say), then $Q'O = B_1'O \cdot OB_1/OQ = b/\mu$. We then have the following equations :

$$\left.\begin{array}{l} (a)\ Q'P\ ;\ y = x\tan(90^\circ - \alpha) - b/\mu, \text{ i.e. } x/a - y/b = 1/\mu \\ \quad\ QP\ ;\ y = x\tan(90^\circ + \alpha) + \mu b, \text{ i.e. } x/a + y/b = \mu \end{array}\right\} \quad \ldots\ldots(3)$$

$$(b)\ AQ,\ A'Q' \text{ respectively ;} \quad x/a + y/b\mu = 1, \quad -x/a - \mu y/b = 1 \quad \ldots\ldots(4)$$

$$A'Q,\ AQ' \text{ respectively ;} \quad -x/a + y/b\mu = 1, \quad x/a - \mu y/b = 1 \quad \ldots\ldots(5)$$

$$(c)\ AQ,\ A'Q_1 \text{ respectively ;} \quad x/a + y/b\mu = 1, \quad -x/a + \mu y/b = 1 \quad \ldots\ldots(6)$$

$$A'Q,\ AQ_1 \text{ respectively ;} \quad -x/a + y/b\mu = 1, \quad x/a + \mu y/b = 1 \quad \ldots\ldots(7)$$

Eliminating μ between the two equations in each pair, we get (1) as the locus of P, R, R', from (3), (4), (5) ; and (2) as the locus of S, S', from (6), (7).

(*d*) This is clearly the converse of (*b*) but can be obtained independently by writing R on (1) as $(a\cosh\mu,\ b\sinh\mu)$, so that RA, RA' are respectively

$$y/(x-a) = b\sinh\mu/\{a(\cosh\mu - 1)\} = (b/a)\coth(\mu/2) \quad \ldots\ldots(8)$$

$$y/(x+a) = b\sinh\mu/\{a(\cosh\mu + 1)\} = (b/a)\tanh(\mu/2) \quad \ldots\ldots(9)$$

Thus Q, Q' are respectively $(0, -b\coth(\mu/2))$, $(0, b\tanh(\mu/2))$ and so $QO \cdot OQ' = b^2 = B_1'O \cdot OB_1$, i.e. Q, B_1, Q', B_1' are concyclic.

(*e*) This is the immediate consequence of (*d*). For, since QQ' is the perpendicular bisector of B_1B_1', it is the diameter of Σ and so perpendicular lines through Q, Q' must meet on Σ.

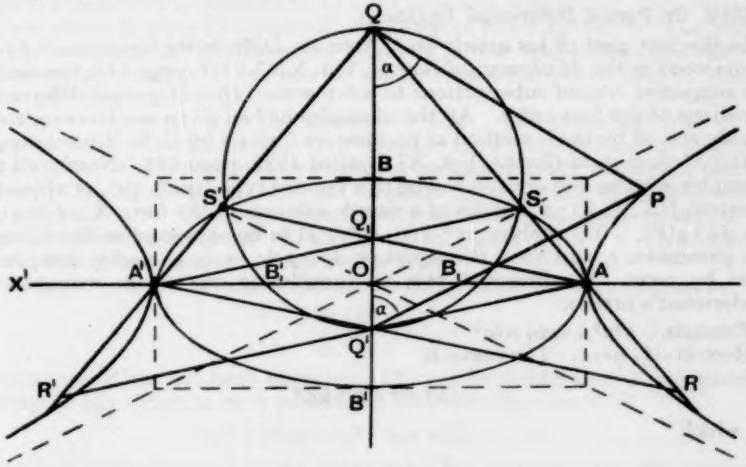

FIG. 1.

It will be readily seen that if we arrange in different ways the equations of the two planes whose line of intersection is a generating line of the hyperboloid of one sheet, viz.

$$x^2/a^2 - y^2/b^2 + z^2/c^2 = 1 \quad (A)\ ;\ \text{or } x^2/a^2 + y^2/b^2 - z^2/c^2 = 1 \quad (B)$$

and then put $z=0$, we get (3), (4), (5) from (*A*) and (6), (7) from (*B*). Thus, for instance, (4) is $z=0$ in $z/c+y/b=\mu(1-x/a)$, $\mu(z/c-y/b)=1+x/a$, with similar adjustments for the other cases.

The hyperbolic paraboloid given by $x/a-y/b=2\mu$, $x/a+y/b=z/\mu$ clearly does not, with $z=0$, yield comparable results for a parabola. However, we do have a parabola theorem and it is even simpler than the above ones but it does not involve circles. It is :

Let OM, ON be two lines making equal angles θ with the axes OX, OY respectively. Let AN be parallel to OY meeting ON at N. Then, if NM be parallel to OX meeting OM at M, the locus of M is a parabola.

For $NA=a\cot\theta$, so NM is $y=a\cot\theta$. OM is $y=x\tan\theta$. Hence $y^2=ax$.

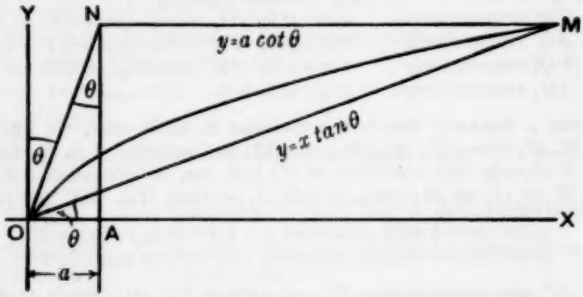

FIG. 2.

34, Stonehill Road, S.W.14. A. J. CARR

2689. On Partial Differential Equations.

In the first part of his article on "*Notes on Differential Equations*" by F. Underwood in the *Mathematical Gazette*, Vol. XXXVIII, page 175, the author has suggested certain substitutions to solve some types of partial differential equations of the first order. All the examples he has given are however more readily solved by the "method of parameters" given by C. N. Srinivasiengar in the *Mathematical Gazette*, Vol. XIV, April 1929, page 423. Nearly all the examples in these two articles belong to a general type $f(px^m, qy^n, z)$ where the equation $f(X, Y, Z)=0$ admits of a simple solution in the form $X=\phi(t)\chi(Z)$, $Y=\psi(t)\chi(Z)$. This enables $px^m/\chi(z)$, $qy^n/\chi(z)$ to be expressed as functions of the parameter t, and then the equation $dz=p\,dx+q\,dy$ is readily integrated term by term. We illustrate this by considering one of the examples in Underwood's article.

Example : $x^2p^2+ayzq=bz^2$

Here $m=1$, $n=1$. The curve is

$$X^2+aYZ=bZ^2$$

for which

$$\frac{X}{Z}=\frac{xp}{z}=\sqrt{b}\cos\alpha$$

$$\frac{Y}{Z}=\frac{yq}{z}=\frac{b}{a}\sin^2\alpha$$

$$\therefore\ dz=p\,dx+q\,dy$$

becomes $$\frac{dz}{z}=\sqrt{b}\cos\alpha\frac{dx}{x}+\frac{b}{a}\sin^2\alpha\frac{dy}{y}$$

$$\therefore \log cz=\sqrt{b}\cos\alpha\log x+\frac{b}{a}\sin^2\alpha\log y$$

is the complete integral where c and α are arbitrary constants.

We give a further example to illustrate the power of the method.

To solve $$(x^2p^2z^6)^3+(y^7q^5z^{10})^3=3ax^3y^7p^3q^5z^{16}.$$

This is based on the curve $X^3+Y^3=3aXY$.

Hence put

$$xpz^2=\left(\frac{3at}{1+t^3}\right)^{\frac{1}{3}}$$

$$y^{\frac{7}{5}}qz^3=\left(\frac{3at^2}{1+t^3}\right)^{\frac{1}{5}}.$$

This leads to $$z^2dz=\left(\frac{3at}{1+t^3}\right)^{\frac{1}{3}}\frac{dx}{x}+\left(\frac{3at^2}{1+t^3}\right)^{\frac{1}{5}}\frac{dy}{y^{\frac{7}{5}}}$$

$$\therefore \frac{z^3}{3}+b=\left(\frac{3at}{1+t^3}\right)^{\frac{1}{3}}\log x-\frac{5}{2}\left(\frac{3at^2}{1+t^3}\right)^{\frac{1}{5}}y^{-\frac{2}{5}}$$

where t and b are variable parameters.

Central College, Bangalore. (Miss) M. N. RAJALAKSHAMMA

2690. True or False.

Is it possible for a theorem to be both true and false at the same time?

In a recent London University Special Degree Examination, the last part of one question reads " If two conics intersect in four real points, and the tangents are perpendicular at each of the intersections, prove that in general these points are concyclic ".

We first " prove " that the theorem is true. The preceding part of the question deals with the harmonic envelope Φ of two conics (the envelope of a variable line which meets the two conics in harmonically conjugate pairs of points). It may be shown that the tangents to each conic at an intersection are tangents to Φ. In the last part of the question the tangents at each of the four intersections are perpendicular, so these points lie on the director circle of Φ. Hence the theorem is apparently true.

We now " prove " that the theorem is false. Let one of the conics be

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}=1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where α and β are not both negative. (It may be shown that it is impossible for one of the conics to be a parabola.) Let the other conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Then it is easily shown that if a point (x, y) lies on each conic, the necessary and sufficient condition that the tangents at (x, y) are perpendicular is

$$a\beta x^2+h(\alpha+\beta)xy+b\alpha y^2+g\beta x+f\alpha y=0 \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

Since (3) must be satisfied at each of the four points of intersection of (1) and (2), (3) must represent a conic of the pencil determined by (1) and (2), and so, for some values of λ and μ, we must have

$$\begin{aligned} a\beta &= \lambda/\alpha + a\mu \\ h(\alpha+\beta) &= 2h\mu \\ b\alpha &= \lambda/\beta + b\mu \\ g\beta &= 2g\mu \\ f\alpha &= 2f\mu \\ 0 &= -\lambda + c\mu. \end{aligned}$$

If we assume that α and β satisfy no special conditions, it is clear that of the three coefficients f, g and h, at least two must be zero. We investigate the different possibilities in turn.

(i) $g=h=0$.

It is easily shown that (2) must take the form

$$\frac{x^2}{2\beta-\alpha}+\frac{y^2}{\beta}+2fy+1=0,$$

where f is arbitrary. It may also be shown that if α, β and f satisfy certain rather complicated inequality conditions, the four points of intersection are all real.

(ii) $f=h=0$.

This is very similar to (i). (2) must take the form

$$\frac{x^2}{\alpha}+\frac{y^2}{2\alpha-\beta}+2gx+1=0,$$

where g is arbitrary.

(iii) $f=g=0$.

It is easily shown that (2) must then take the form

$$\frac{x^2}{\alpha}+2hxy-\frac{y^2}{\beta}+\frac{\beta-\alpha}{\alpha+\beta}=0,$$

where h is arbitrary. As before, the four points of intersection can be all real.

(iv) $f=g=h=0$.

It is easily shown that (2) must then take the form

$$\frac{x^2}{\alpha-\lambda}+\frac{y^2}{\beta-\lambda}=1,$$

where λ is arbitrary. As before, the four points of intersection can be all real. This is the well-known case where the conics are confocal.

Now it is easily shown that in cases (i), (ii) and (iv) the pencil of conics determined by (1) and (2) contains a circle, so the four intersections are concyclic. In case (iii), however, except in the special case where $h=0$, there is clearly no conic of the pencil which is a circle, so the four intersections are not concyclic. Since (iii) is not a special case of (i), (ii) or (iv), the theorem is apparently false.

In order to resolve the paradox, we must examine case (iii) more carefully, from the point of view of our first method. The equation of the harmonic envelope Φ of (1) and (2) is then

$$l^2-m^2+h(\alpha+\beta)lm=0,$$

so Φ degenerates into a pair of points at infinity in perpendicular directions. In such a case, *every* finite point P of the plane satisfies the condition that the tangents to Φ from P are perpendicular, so we can no longer conclude that the four intersections are concyclic.

From this point of view, it seems that (iii) might be regarded as a special case (since Φ degenerates) and that therefore the theorem is true in general.

On the other hand, when we say that a theorem is true in general, we mean that the objects mentioned *in the statement of the theorem* satisfy no special conditions, other than those specified. Now Φ is not mentioned in the statement of our theorem, so the fact that Φ degenerates does not necessarily imply that (iii) is a special case. Our second method shows that the specified conditions, namely the fact that the tangents are perpendicular at each of the four intersections, give rise to four different possibilities, and in each of the four cases no further special conditions are satisfied. Hence (iii) is not a special case, and so *the theorem is false*.

The moral of this is that, to avoid ambiguity, it is desirable to say exactly what is meant by the term " in general ".

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2691. Use of a " D-backwards " operator.

Usually the symbol for an operator precedes that for the function on which it operates. In particular this is so for d/dx and its abbreviated form D. Here we introduce a companion operator $\mathcal{D}$ (which may be called *eed*) to indicate differentiation of the function *preceding* it.

With this notation the rule for differentiating a product may be expressed as

$$D(uv) = u\mathcal{D}v + uDv = u(\mathcal{D} + D)v.$$

If we rewrite this as $1 \,.\, (\mathcal{D} + D) \,.\, uv = u(\mathcal{D} + D)v = uv \,.\, (\mathcal{D} + D) \,.\, 1$ we see that $\mathcal{D} + D$, in contrast with D or $\mathcal{D}$ separately, commutes with any function of x.

If we accept that u, followed by m $\mathcal{D}$'s and n D's in any order, followed by v stands for $u_m v_n$, where subscripts denote number of differentiations, Leibniz' theorem can be written down thus :

$$D^n(uv) = (\mathcal{D} + D)^n uv = u(\mathcal{D} + D)^n v$$

$$= u\left\{\mathcal{D}^n + \binom{n}{1}\mathcal{D}^{n-1}D + \binom{n}{2}\mathcal{D}^{n-2}D^2 + \ldots + D^n\right\} v$$

$$= u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \ldots + uv_n$$

More generally we have

$$f(D)uv = 1 \,.\, f(\mathcal{D} + D)uv = uf(\mathcal{D} + D)v$$

for any polynomial f. With $u = e^{kx}$ (k constant), this gives

$$f(D)e^{kx}v = e^{kx}f(\mathcal{D} + D)v.$$

We may replace $\mathcal{D}$ by k when it operates on e^{kx} thus obtaining the result

$$f(D)e^{kx}v = e^{kx}f(k + D)v.$$

Turning now to the use of inverse operators in finding particular integrals, we have in the following a plausible alternative to the customary excursion into imaginaries :

$$\frac{1}{D^2 + a^2}\cos ax = 1\,\frac{1}{(\mathcal{D} + D)^2 + a^2}\cos ax \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= 1\,\frac{1}{\mathcal{D}^2 + 2\mathcal{D}D + D^2 + a^2}\cos ax$$

$$= 1\,\frac{1}{\mathcal{D}(\mathcal{D} + 2D)}\cos ax, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

replacing D^2 acting on $\cos ax$ by $-a^2$.

$$= 1\,\frac{1}{2\mathcal{D}D}\cos ax, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

replacing $\mathcal{D}$ in bracket, acting on 1, by 0.

$$= \int \tfrac{1}{2}\, dx \int \cos ax\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$= \frac{x \sin ax}{2a}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

the correct result.

A possible method of interpreting and justifying this procedure is suggested by the observation that if $f(\mathcal{D}, D)$ is a polynomial in $\mathcal{D}$ and D, $u(x) f(\mathcal{D}, D) v(x)$ is the same as the result of putting $y = x$ in

$$f(D_x, D_y) u(x) v(y), \quad \text{where } D_x \equiv \frac{\partial}{\partial x}, \; D_y \equiv \frac{\partial}{\partial y}.$$

To deal with (1), we first note that if z is a function of x and y,

$$dz = D_x z\, dx + D_y z\, dy$$

and along $y = x$, $dx = dy$. It follows that along $y = x$,

$$Dz(x, x) = D_x z + D_y z = (D_x + D_y) z,$$

and hence more generally

$$f(D) z(x, x) = f(D_x + D_y) z,$$

for polynomial f. We can go further and say that

$$\frac{1}{f(D)}\, z(x, x) = \left\{ \frac{1}{f(D_x + D_y)}\, z \right\}_{y = x},$$

meaning that the functions of x satisfying $f(D) w = z$ are precisely the functions obtained by putting $y = x$ in solutions of the partial differential equation $f(D_x + D_y) w = z$. If $z = u(x) v(y)$ this gives an interpretation of the equation

$$\frac{1}{f(D)}\, uv = u \frac{1}{f(\mathcal{D} + D)}\, v,$$

and in particular therefore of (1) above.

On this interpretation the steps (2), (3), (4) and (5) are essentially a procedure obtaining a particular integral for

$$(D_x^2 + 2D_x D_y + D_y^2 + a^2) z = \cos ay \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and putting $y = x$ in the result. Step (2), of replacing D^2 by $-a^2$ corresponds to replacing D_y^2 by $-a^2$, a step permissible if and only if z satisfies, in addition to (6), the condition $(D_y^2 + a^2) z = 0$. The appearance of this condition need cause no surprise, for the replacement of D^2 by $-a^2$ in $\dfrac{1}{f(D)} \cos ax$ gives correctly only that solution, if any, of the form $A \cos ax + B \sin ax$. The step (3) is accompanied by the imposition of a further condition. Corresponding to the expression (4), the interpretation has

$$\int \tfrac{1}{2}\, dx \int \cos ay\, dy = \frac{x}{2} \frac{\sin ay}{a}.$$

Putting $y = x$ now gives the desired result. Note that $\dfrac{\sin ay}{a} + c$ would not be a permissible integral of $\cos ay$, as

$$z = \frac{x}{2} \left(\frac{\sin ay}{a} + c \right)$$

does not satisfy $(D_y^2 + a^2) z = 0$ unless $c = 0$.

The type of interpretation and justification here could form a basis for rules for employing expressions of the form $u\dfrac{g(D, D)}{f(D, D)}v$, covering for example,

$$\int x^5 \sin x\, dx = \sin x \frac{1}{D+D} x^5$$

$$= \sin x \frac{D-D}{D^2-D^2} x^5$$

$$= \sin x \frac{D-D}{1+D^2} x^5$$

$$= (\sin xD - \cos x)(1 - D^2 + D^4)x^5$$

$$= \sin x(5x^4 - 60x^2 + 120) - \cos x(x^5 - 20x^3 + 120x).$$

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2692. Expansion of a Certain Infinite Product.

Suppose that the product $\prod_{n=1}^{\infty}(1+\cos 4^n\theta)$ is expanded as the formal series

$$1 + \alpha_1 \cos p_1\theta + \alpha_2 \cos p_2\theta + \ldots,$$

where the integers p_n are in ascending order. The terms of this series can easily be calculated by expansion of successive partial products. Thus

$$\prod_{n=1}^{2} = (1+\cos 4\theta)(1+\cos 16\theta) = 1 + \cos 4\theta + \tfrac{1}{2}\cos 12\theta + \cos 16\theta + \tfrac{1}{2}\cos 20\theta,$$

$$\prod_{n=1}^{3} = (1+\cos 4\theta)(1+\cos 16\theta)(1+\cos 64\theta)$$

$$= \prod_{n=1}^{2} + \tfrac{1}{4}\cos 44\theta + \tfrac{1}{2}\cos 48\theta + \tfrac{1}{4}\cos 52\theta + \tfrac{1}{2}\cos 60\theta + \cos 64\theta + \tfrac{1}{2}\cos 68\theta + \ldots + \tfrac{1}{4}\cos 84\theta,$$

and so on. The question arises: given n, what is $\alpha_n \cos p_n\theta$? We shall show that any integer n can be expressed in the form

$$n = 3^k + a_1 3^{k-1} + a_2 3^{k-2} + \ldots + a_k, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where k is an integer and the a's take the values $0, +1$ or -1, and that we then have

$$\alpha_n \cos p_n\theta = \frac{1}{2^k}\cos(4^{k+1} + a_1 4^k + a_2 4^{k-1} + \ldots + a_k 4)\theta. \qquad \ldots\ldots\ldots\ldots(2)$$

(1) follows by induction, for suppose that (1) is true for all n up to and including $n = N_m = 3^m + 3^{m-1} + \ldots + 1$. Then the integers from $N_m + 1$ to N_{m+1} are

$$3^{m+1} - N_m,\ 3^{m+1} - (N_m - 1),\ 3^{m+1} - (N_m - 2), \ldots 3^{m+1},\ 3^{m+1} + 1, \ldots 3^{m+1} + N_m, \qquad \ldots\ldots\ldots\ldots(3)$$

and these are all of the form (1). The inductive argument is complete on verifying (1) for $n = 1\,(m = 0)$.

We next make the inductive hypothesis that

$$\prod_{n=1}^{m+1} = 1 + \sum_{n=1}^{N_m} \alpha_n \cos p_n\theta,$$

where the terms in the sum are given by (2). Since

$$\prod_{n=1}^{m+2} = \prod_{n=1}^{m+1} + \cos 4^{m+2}\theta \left(1 + \sum_{n=1}^{N_m} \alpha_n \cos p_n\theta\right),$$

and since $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$, the terms $\alpha_n \cos p_n\theta$, $n = N_m + 1, \ldots N_{m+1}$, are

$$\tfrac{1}{2}\alpha_{N_m}\cos(4^{m+2} - p_{N_m})\theta,\ \tfrac{1}{2}\alpha_{N_m-1}\cos(4^{m+2} - p_{N_m-1})\theta,\ \tfrac{1}{2}\alpha_{N_m-2}\cos(4^{m+2} - p_{N_m-2}\theta),$$
$$\ldots\cos 4^{m+2}\theta,\ \tfrac{1}{2}\alpha_1\cos(4^{m+2} + p_1)\theta, \ldots \tfrac{1}{2}\alpha_{N_m}\cos(4^{m+2} + p_{N_m})\theta, \quad \ldots\ldots\ldots(4)$$

and on comparing (4) with (3) we see that (2) is established for $n = N_m + 1, \ldots N_{m+1}$. The inductive argument is again complete on verifying (2) for $m = 0$.

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2693. Note on some quadrature formulae.

It is well known that of the quadrature formulae, Simpson's one-third rule, the three-eighths rule and the Weddle's rule, the third is the most accurate and the second is the least accurate. For a discussion of this, see J. B. Scarborough [2]. In this note it is shown that Hardy's formula, widely used in actuarial work, is better than either of the Simpson's rules and less accurate than Weddle's rule. Comparison between Simpson's one-third rule and Boole's formula surprisingly shows that the latter is more accurate.

The smallest number of subintervals for which a comparison among the Simpson's rules, Hardy's formula and Weddle's rule can be made is six. We shall consider intervals of width $6h$. Let $f(x)$ be the integrand and

$$F(x) + C = \int_k^x f(x)dx.$$

Then the true value of the integral denoted by I is

$$I = \int_{a-3h}^{a+3h} f(x)dx = F(a+3h) - F(a-3h). \quad \ldots\ldots\ldots(1)$$

By Hardy's formulae its value is

$$I_H = h[2{\cdot}2\,f(a) + 1{\cdot}62\{f(a-2h) + f(a+2h)\} + {\cdot}28\{f(a-3h) + f(a+3h)\}]. \ldots(2)$$

Following Scarborough, making the assumptions given therein, and replacing the various functions in (1) and (2) by the corresponding Taylor expansions and remembering that

$$F'(x) = f(x),\ F''(x) = f'(x) \text{ etc., we get}$$

$$I = 6h\,f(a) + 9h^2 f''(a) + \tfrac{81}{20}h^3 f^{iv}(a) + \tfrac{243}{280}h^7 f^{vi}(a) + \ldots \quad \ldots\ldots\ldots(3)$$

and $$I_H = 6h\,f(a) + 9h^2 f^{ii}(a) + \tfrac{81}{20}h^5 f^{iv}(a) + \tfrac{171}{200}h^7 f^{vi}(a) + \ldots \quad \ldots\ldots\ldots(4)$$

So the error E_H (numerically) is

$$E_H = \tfrac{9}{700}h^7 f^{vi}(a) + \ldots \quad \ldots\ldots\ldots(5)$$

The corresponding errors for the other rules denoted by $E_{1/3}$, $E_{3/8}$, E_W (Scarborough) are

$$E_{1/3} = \frac{h^5}{30} f^{iv}(a) + \frac{17}{420} h^7 f^{vi}(a) + \ldots \quad \ldots\ldots\ldots(6)$$

$$E_{3/8} = \frac{3h^5}{30} f^{iv}(a) + \frac{53}{560} h^7 f^{vi}(a) + \dots \quad \dots\dots(7)$$

and

$$E_W = \frac{h^7}{140} f^{vi}(a) + \dots \quad \dots\dots(8)$$

It follows from (6), (7), (8) Weddle's rule is the best and next to that in accuracy is Hardy's formula.

Now Boole's formula is given by [1]

$$I_B = \int_{a-2h}^{a+2h} f(x)dx = \frac{4h}{90}[7\{f(a-2h) + f(a+2h)\} + 12f(a) + 32\{f(a-h) + f(a+h)\}].$$

In order that a comparison may be made between this formula and the one-third rule, the smallest number of subintervals must be four. We shall accordingly consider intervals of width $4h$. Arguing as before, the true value of the integral becomes

$$I = 4h\left[f(a) + \frac{4}{3!}h^2 f''(a) + \frac{16}{5!}h^4 f^{iv}(a) + \frac{64}{7!}h^6 f^{vi}(a) + \dots\right]$$

Now

$$I_B = 4h\left[f(a) + \frac{4}{3!}h^2 f''(a) + \frac{16}{5!}h^4 f^{iv}(a) + \frac{224}{3}\frac{h^6}{7!} f^{vi}(a) + \dots\right] \quad \dots\dots(9)$$

and

$$I_{1/3} = 4h\left[f(a) + \frac{4}{3!}h^2 f''(a) + \frac{50}{3}\frac{h^4}{5!} f^{iv}(a) + \frac{238}{3}\frac{h^6}{7!} f^{vi}(a) + \dots\right] \quad \dots\dots(10)$$

The errors are therefore

$$E_B = -\frac{4}{3}\left[32\frac{h^7}{7!} f^{vi}(a) + \dots\right] \quad \dots\dots(11)$$

and

$$E_{1/3} = -\frac{4}{3}\left[2\frac{h^5}{5!} f^{iv}(a) + 46 \cdot \frac{h^7}{7!} f^{vi}(a) + \dots\right] \quad \dots\dots(12)$$

from which it follows that Boole's formula is better than the one-third rule.

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REFERENCES

1. Jordan, C., (1950), *Calculus of Finite Differences*, Chelsea Publishing Co., New York.

2. Scarborough, J. B., (1949). *Numerical Mathematical Analysis*. The Johns Hopkins press, pages 154–156.

2694. A note on Taylor's theorem.

Among the many formulae connected with Taylor's theorem given in text-books on the differential calculus, I have not been able to find the following result, which gives an explicit formula for the remainder after n terms in terms of a repeated derivative :

$$(1) \quad \psi(b) = \psi(x) + (b-x)\psi'(x) + \frac{(b-x)^2}{2!}\psi''(x) + \dots + \frac{(b-x)^{n-1}}{(n-1)!}\psi^{(n-1)}(x) + \frac{(b-x)^n}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{\psi(b) - \psi(x)}{b-x}\right\}.$$

The formula is, of course, a simple consequence of Leibniz's theorem on repeated differentiation.

If we combine (1) with Lagrange's form of Taylor's theorem, we get immediately

$$\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{\psi(b)-\psi(x)}{b-x}\right\}=\frac{1}{n}\psi^{(n)}(\xi), \tag{2}$$

where ξ lies between x and b. The expression in brackets on the left is the slope of the chord joining the two points of the curve $y=\psi(x)$ at x, b, so that this relation may be regarded as a direct generalization of the Mean Value Theorem in its geometrical form, i.e. that the slope of the chord joining two points of the curve $y=\psi(x)$ is equal to the slope of the tangent to the curve at some intermediate point.

It does not seem possible to derive (2) directly,* assuming only the existence of $\psi^{(n)}(x)$. Given the continuity of $\psi^{(n)}(x)$, however, we can obtain the usual integral form of the remainder, that is, the formula

$$\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{\psi(b)-\psi(x)}{b-x}\right\}=\frac{1}{(b-x)^n}\int_x^b(b-u)^{n-1}\psi^{(n)}(u)\,du. \tag{3}$$

This may be verified by evaluating the expression on the left by Leibniz's theorem and that on the right by integration by parts. It may also be proved in two or three lines by induction. There is also a third proof, rather highbrow, but more direct. Let $b-x=t$, $\psi(b)-\psi(x)=\phi(t)$. Then (3) becomes

$$\frac{d^{n-1}}{dt^{n-1}}\left\{\frac{\phi(t)}{t}\right\}=\frac{1}{t^n}\int_0^t u^{n-1}\phi^n(u)\,du, \tag{4}$$

and it is enough to prove this. We observe first that for any f

$$\frac{d^{n-1}}{dt^{n-1}}f(t)=\frac{1}{t^{n-1}}\left[\frac{d^{n-1}}{da^{n-1}}f(at)\right]_{a=1},$$

where on the right a is put equal to 1 after differentiation. Since $\phi(0)=0$,

$$\frac{d^{n-1}}{da^{n-1}}\left\{\frac{\phi(at)}{at}\right\}=\frac{d^{n-1}}{da^{n-1}}\left\{\frac{1}{at}\int_0^{at}\phi'(u)\,du\right\}=\frac{d^{n-1}}{da^{n-1}}\left\{\frac{1}{t}\int_0^t\phi'(au)\,du\right\}$$

$$=\frac{1}{t}\int_0^t u^{n-1}\phi^{(n)}(au)\,du,$$

and the result (4) follows.

Combining (1) and (3), we thus obtain a proof of Taylor's theorem, with the remainder in the form of an integral, using Leibniz's theorem, the fundamental theorem of the integral calculus, and differentiation under the sign of integration. The material is at least novel.

The expression

$$F(x)=\psi(b)-\psi(x)-(b-x)\psi'(x)-\ldots-\frac{(b-x)^{n-1}}{(n-1)!}\psi^{(n-1)}(x) \tag{5}$$

$$=\frac{(b-x)^n}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{\psi(b)-\psi(x)}{b-x}\right\}$$

which occurs in (1) occurs also in the standard proof of Schlömilch's form of Taylor's theorem.† It may be worth while to comment here that the proof of this form of Taylor's theorem given in most English text-books is merely a repetition of the argument used to prove Cauchy's Mean Value Theorem. A direct appeal to the latter theorem, however, gives a proof which is a good deal

* That is, without expanding the left-hand side by Leibniz's theorem.

† See e.g. Phillips, *A Course of Analysis* (Cambridge, 1930), p. 105.

more succinct.* Let $F(x)$ be defined by (5), and let $G(x)=(b-x)^p$, where p is a positive integer. Then, by Cauchy's Mean Value Theorem,

$$\frac{F(a)}{(b-a)^p}=\frac{F(b)-F(a)}{G(b)-G(a)}=\frac{F'(\xi)}{G'(\xi)}=\frac{(b-\xi)^{n-p}\psi^{(n)}(\xi)}{p(n-1)!}$$

for some ξ between a and b. Writing $\xi=a+\theta(b-a)$, where $0<\theta<1$, we have

$$F(a)=\frac{(1-\theta)^{n-p}(b-a)^n\psi^{(n)}\{a+\theta(b-a)\}}{p(n-1)!}.$$

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2695. A generalisation of Taylor's theorem.

The generalisation of the Mean-Value Theorem given in the previous note is a particular case of the following generalisation of Taylor's theorem: If $f(t)$ is differentiable $n+r+1$ times for $a\leqslant t\leqslant x$, then there is a ξ between a and x such that

$$\frac{d^r}{dx^r}\left\{\left[f(x)-f(a)-(x-a)f'(a)-\frac{(x-a)^2}{2!}f''(a)-\ldots\right.\right.$$
$$\left.\left.-\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)\right]\Big/(x-a)^n\right\}=\frac{r!}{(n+r)!}f^{n+r}(\xi).$$

We commence by proving that

$$(x-a)^r D^{n+r}(x-a)^n\,\psi(x)=D^n\{(x-a)^{n+r}D^r\psi(x)\},$$

for any sufficiently often differentiable $\psi(x)$.

We proceed by induction over n; if this formula holds for $n=k$ then, with $h(x)$ standing for $(x-a)^{k+r}D^r\psi(x)$, and $H(x)$ for $D^{k+r}(x-a)^k\psi(x)$, we have

$$\begin{aligned}D^{k+1}\{(x-a)^{k+r+1}D^r\psi(x)\}&=D^{k+1}(x-a)h(x)\\&=(x-a)D^{k+1}h(x)+(k+1)D^kh(x)\\&=(x-a)\,D(x-a)^r\,H(x)+(k+1)(x-a)^r\,H(x)\\&=(k+r+1)(x-a)^r\,H(x)+(x-a)^{r+1}\,DH(x);\end{aligned}$$

but
$$\begin{aligned}D^{k+r+1}(x-a)^{k+1}\,\psi(x)&=D^{k+r+1}(x-a)\,.\,(x-a)^k\psi(x)\\&=(x-a)\,DH(x)+(k+r+1)\,H(x),\end{aligned}$$

which proves the formula for $n=k+1$, and since it is obviously true with $n=0$, it follows that the formula is true for all values of n.

Taking $\phi(x)=(x-a)^n\psi(x)$ we find

$$D^n\left\{(x-a)^{n+r}D^r\frac{\phi(x)}{(x-a)^n}\right\}=(x-a)^rD^{n+r}\phi(x).$$

By Leibniz's theorem there are constants $C_0, C_1, C_2,\ldots, C_r$ such that

$$\begin{aligned}&(x-a)^{n+r}\,D^r\frac{\phi(x)}{(x-a)^n}\\&\quad=C_0\phi(x)+C_1(x-a)\phi'(x)+C_2(x-a)^2\phi''(x)x)+\ldots+C_r(x-a)^r\phi^r(x)\end{aligned}$$

and so there are constants $k_0, k_1,\ldots, k_r$ such that

$$\begin{aligned}&D^p\left\{(x-a)^{n+r}D^r\frac{\phi(x)}{(x-a)^n}\right\}\\&\quad=k_0\phi^p(x)+k_1(x-a)\phi^{p+1}(x)+k_2(x-a)^2\phi^{p+2}(x)+\ldots+k_r(x-a)^r\phi^{p+r}(x).\end{aligned}$$

* A similar proof of a still more general form of Taylor's theorem is given in Chaundy, *The Differential Calculus* (Oxford, 1935), p. 110, the deduction of Schlömilch's form there being relegated to an exercise (p. 135, Ex. 40).

It follows that if

$$\phi(x) = f(x) - f(a) - (x-a)f'(a) - \ldots - \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

so that $\phi(a) = \phi'(a) = \ldots = \phi^{n-1}(a) = 0$ and $\phi^n(x) = f^n(x)$,

and if

$$F(x) = (x-a)^{n+r} D^r \frac{\phi(x)}{(x-a)^n},$$

then $F(a) = F'(a) = \ldots = F^{n-1}(a) = 0$ and $F^n(x) = (x-a)^r f^{n+r}(x)$

so that, by repeated application of Cauchy's formula, there exists ξ such that

$$\frac{F(x)}{(x-a)^{n+r}/(n+r)!} = \frac{F^n(\xi)}{(\xi-a)^r/r!} = r!\, f^{n+r}(\xi)$$

which completes the proof. R. L. GOODSTEIN

2696. A Method of Summing Rational Integral Functions.

If we wish to obtain a formula for the sum of any number of terms (starting with the first) of a rational integral function, we may sum each power independently, employing Bernoulli's Numbers, or we may transform the powers into factorial powers and use the methods of finite differences. If the final formula is required in ordinary powers, the alternative method shown in this Note may be of interest.

The notation used will be :

$$\nabla u_x = u_x - u_{x-1}.$$

$n_{(t)}$ = the binomial coefficient $\binom{n}{t}$.

$$\Sigma u_x = u_1 + u_2 + \ldots + u_x.$$

We thus have :

$$\begin{aligned}\nabla x^n &= x^n - (x-1)^n \\ &= x^n - \{x^n - n \,.\, x^{n-1} + n_{(2)}x^{n-2} - n_{(3)}x^{n-3} + \ldots\} \\ &= n \,.\, x^{n-1} - n_{(2)}x^{n-2} + n_{(3)}x^{n-3} - \ldots \\ &= \sum_{t=1}^{n} n_{(t)} x^{n-t}(-1)^{t+1}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}\end{aligned}$$

eg., $\nabla x^5 = 5x^4 - 10x^3 + 10x^2 - 5x + 1.$

Using the definitions given above, we have :

$$\begin{aligned}\Sigma \nabla u_x &= (u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \ldots + (u_{x-1} - u_{x-2}) + (u_x - u_{x-1}) \\ &= u_x - u_0.\end{aligned}$$

Whence, since $[x^n]_{x=0} = 0$ we have :

$$\Sigma \nabla x^n = x^n.$$

From (i) above we therefore derive :

$$\Sigma \left\{ \sum_{t=1}^{n} n_{(t)} x^{n-t}(-1)^{t+1} \right\} = x^n. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

e.g. $\Sigma\{5x^4 - 10x^3 + 10x^2 - 5x + 1\} = x^5.$

We may use this result to obtain the summation of any rational integral function. Let us consider the example :

$$\Sigma\{2x^3 + 6x^2 - 9x + 1\}$$

Let S denote this sum, so that

$$\Sigma\{2x^3+6x^2-9x+1\}=S \qquad \text{.....(iii)}$$

From (ii), with $n=4$, we obtain, after dividing by 4,

$$\Sigma\{x^3-\tfrac{3}{2}x^2+x-\tfrac{1}{4}\}=\frac{x^4}{4}$$

and so

$$\Sigma\{2x^3-3x^2+2x-\tfrac{1}{2}\}=\frac{x^4}{2} \qquad \text{.....(iv)}$$

Taking (iv) from (iii) gives

$$\Sigma\{9x^2-11x+\tfrac{3}{2}\}=S-\frac{x^4}{2}. \qquad \text{.....(v)}$$

From (ii) again, with $n=3$, after dividing by 3 and multiplying by 9, we obtain

$$\Sigma\{9x^2-9x+3\}=\tfrac{9}{3}x^3$$

and subtracting this from (v) gives

$$\Sigma\{-2x-\tfrac{3}{2}\}=S-\frac{x^4}{2}-3x^3. \qquad \text{.....(vi)}$$

From (ii) yet again, with $n=2$,

$$\Sigma(-2x+1)=-x^2$$

and subtracting this from (vi) gives

$$\Sigma(-\tfrac{5}{2})=S-\frac{x^4}{2}-3x^3+x^2$$

whence $S=\dfrac{x^4}{2}+3x^3-x^2-\tfrac{5}{2}x.$

A systematic way of setting out this calculation is given below.

		$2+6-9+1$
$4-6+4-1$	$1-\frac{3}{2}+1-\frac{1}{2}$	$2-3+2-(\frac{1}{2})$
		$9-11+\frac{3}{2}$
$3-3+1$	$1-1+\frac{1}{2}$	$9-9+(3)$
		$-2-\frac{3}{2}$
$2-1$	$1-\frac{1}{2}$	$-2+(1)$
		$-(\frac{5}{2})$

$$\Sigma\{2x^3+6x^2-9x+1\}=\frac{x^4}{2}+3x^3-x^2-\tfrac{5}{2}x.$$

$$=\frac{x}{2}(x^3+6x^2-2x-5).$$

Explanation.

(1) Write down the coefficients of the expression to be summed at the top of the R.H. column.

(2) If the highest power of x in this expression is x^n, write down on the next line, at the top of the L.H. column, the binomial coefficients $(n+1)$, $-(n+1)_{(2)}$, $(n+1)_{(3)}$, $-\ldots$.

(3) On the same line, at the top of the middle column, write the same coefficients, each divided by $n+1$.

(4) In the R.H. column, write down the coefficients of the middle column, each multiplied by the first coefficient of the R.H. column.
(5) Subtract the coefficients in the R.H. column.
(6) On the next line, in the L.H. column, write down the binomial coefficients n, $-n_{(2)}$, $n_{(3)}$, $-\ldots$.
(7) Repeat the above process, until there is only one term in the R.H. column.
(8) If the highest power of x in the expression to be summed is x^n, the highest power of x in the sum will of course be x^{n+1}. The values in brackets in the R.H. column are the required coefficients.

The sign in each case will be the sign of the first coefficient of the set.

Although the explanation is of necessity somewhat lengthy, the execution is simple and speedy. In many cases, after a little facility has been achieved, the middle column may be eliminated.

A second example will demonstrate that the number of coefficients in the R.H. column must only be reduced by one at a time, zero being used for the first coefficient of a set if necessary.

Example (2). Evaluate Σx^5.

$6-15+20-15+6-1$	$1-\frac{5}{2}+\frac{10}{3}-\frac{5}{2}+1-\frac{1}{6}$	$1+0+0+0+0+0$ $1-\frac{5}{2}+\frac{10}{3}-\frac{5}{2}+1-(\frac{1}{6})$
$5-10+10-5+1$		$\frac{5}{2}-\frac{10}{3}+\frac{5}{2}-1+\frac{1}{6}$ $\frac{5}{2}-5+5-\frac{5}{2}+(\frac{1}{6})$
$4-6+4-1$	$1-\frac{3}{2}+1-\frac{1}{4}$	$\frac{5}{3}-\frac{5}{2}+\frac{3}{2}-\frac{1}{3}$ $\frac{5}{3}-\frac{5}{2}+\frac{5}{3}-(\frac{5}{12})$
$3-3+1$		$0-\frac{1}{6}+\frac{1}{12}$ $0-0+(0)$
$2-1$		$-\frac{1}{6}+\frac{1}{12}$ $-\frac{1}{6}+(\frac{1}{12})$
		(0).

From which we see that the coefficients of x^3 and x are zero. We thus have:

$$\Sigma x^5=\frac{x^6}{6}+\frac{x^5}{2}+\frac{5x^4}{12}-\frac{x^2}{12}.$$

The last example suggests that, by employing this method to evaluate the sums of even powers of the integers, we may derive the Bernoulli Numbers. But this method of calculation is not expeditious.

Scole Primary School, Diss, Norfolk. R. V. PARKER

2697. Squash Chances.

In the game of squash a player wins a point if he serves and wins a rally, but not if he wins a rally in which he did not serve ; the winner of a rally serves for the next. The game is won by the first player to obtain nine points (ignoring the possible change which may be made if the score reaches eight-all). The question arises of the advantage given to the player who takes the first service.

In a previous note* I considered the particular case in which the players were of equal ability (by which was meant that in any rally A's chances of winning were $\frac{1}{2}$ and B's were $\frac{1}{2}$), and gave a numerical solution. But now consider the

* Mathematical Gazette, Note 2226.

case in which A and B are of different ability ; in a given rally let A's chances of winning be x and B's be $(1-x)$. At a given stage A is about to serve and needs p points to win the game and B needs q. What are A's chances of winning?

When the score is such that A needs p points to win the game and B needs q, call A's chances of winning the game $a_{p,q}$ if he is serving and $b_{p,q}$ if B is serving. Then

$$\left.\begin{aligned} a_{p,q} &= xa_{p-1,q} + (1-x)b_{p,q}, \\ b_{p,q} &= xa_{p,q} + (1-x)b_{p,q-1}; \end{aligned}\right\} \quad \text{.....(1)}$$

with the end conditions

$$\left.\begin{aligned} a_{0,q} &= 1 \quad (q>0), \\ b_{p,0} &= 0 \quad (p>0). \end{aligned}\right\} \quad \text{.....(2)}$$

The problem is to solve the difference equation (1) explicitly for general $a_{p,q}$ in terms of x, p, q. The best form of solution I am able to find is

$$a_{p,q} = 1 - \sum_{n=1}^{p} \sum_{r=1}^{n} {}^{p}C_{n}\,{}^{q+r-2}C_{r-1}\,\frac{x^{p-n+2r-2}(1-x)^{q+n}}{\{1-x(1-x)\}^{q+r-1}}; \quad \text{.....(3)}$$

but this is clearly not satisfactory as an answer, and I should like to propose the problem to readers of the Gazette—in either the form (1) or the form (3).

Decca Research Laboratories. H. G. ApSimon

2698. Some Algebraic Identities.

This is an attempt to show that some algebraic identities, which are usually solved by cyclic symmetry may be solved in a very elegant and interesting way.

I.

(i) $E \equiv \Sigma bc(b-c)$
(ii) $E \equiv \Sigma a^2(b-c)$
(iii) $E \equiv \Sigma a(c^2-b^2)$.
Each $\equiv -(b-c)(c-a)(a-b)$.

(i), (ii) and (iii) may be shown to be equal by rearranging the terms in (i).

Adding (i) and (ii) $2E = \Sigma(a^2+bc)(b-c)$

Adding to this, (iii),

$$\begin{aligned} 3E &= \Sigma(b-c)(a^2+bc-ac-ab) \\ &= \Sigma(b-c)(a-c)(a-b) \end{aligned}$$

Hence the result.

II.

(i) $E \equiv \Sigma a(b^2+c^2)+3abc$
(ii) $E \equiv \Sigma a^2(b+c)+3abc$
(iii) $E \equiv \Sigma a(b+c)^2-3abc$
(iv) $E \equiv \Sigma bc(b+c)+3abc$
Each $\equiv (a+b+c)(ab+bc+ca)$.

(i), (ii), (iii) and (iv) are all the same expressions because one can be obtained with the help of the other.

Adding (iii) and (iv) we get

$$\begin{aligned} 2E &= \Sigma(b+c)(ab+bc+ca) \\ &= (ab+bc+ca)\{(a+b)+(b+c)+(c+a)\} \end{aligned}$$

Hence the result follows.

III. (i) $E \equiv \Sigma a(b^2+c^2)+2abc$
(ii) $E \equiv \Sigma a^2(b+c)+2abc$
(iii) $E \equiv \Sigma a(b+c)^2-4abc$
(iv) $E \equiv \Sigma bc(b+c)+2abc.$
Each $=(a+b)(b+c)(c+a).$

Since (i), (ii), (iii) and (iv) are different forms of the same expression, adding (ii), (iii) and (iv) we get

$$\begin{aligned} 3E &= \Sigma(b+c)(a^2+bc+ab+ac) \\ &= \Sigma(b+c)(c+a)(a+b) \\ &= 3(a+b)(b+c)(c+a). \end{aligned}$$

B.C. 152 Camp, Belgaum, India. A. K. RAJAGOPAL

2699. A property of pairs of polygons.

In Note 2533 (*Gazette* XXXIX, p. 211) Wright has proved an interesting property of the quadrilateral: Let $A_1A_2A_3A_4$ be a quadrilateral and let $C_1A_1A_2$, $C_2A_2A_3$, $C_3A_4A_3$, $C_4A_1A_4$ be similar triangles, right-angled at the C-points, and constructed to lie outside the quadrilateral; then the diagonals C_1C_3, C_2C_4 of the quadrilateral $C_1C_2C_3C_4$ are equal and the angles between them are twice the acute angles of any of the right-angled triangles.

In a paper (*Mathesis*, 1955, p. 340), I have given a general theorem, which includes not only Wright's theorem but also others mentioned in Note 2533.

R. GOORMAGHTIGH

2700. A tripos question.

The relation (iii) in Prof. Watson's article in the Gazette for December 1955 may well have been discovered quite simply thus.

$$a+b+c=x+y+z=0.$$

Let $ax+by+cz=\lambda$, $ay+bz+cx=\mu$, $az+bx+cy=\nu$. Then

$$\lambda^2-\mu\nu=(a^2-bc)(x^2-yz)+(b^2-ca)(y^2-zx)+(c^2-ab)(z^2-xy).$$

But $a^2-bc=b^2-ca=c^2-ab=-(bc+ca+ab)=\frac{1}{2}(a^2+b^2+c^2)$

and $z^2-xy=x^2-yz=y^2-zx=-(yz+zx+xy)=\frac{1}{2}(x^2+y^2+z^2)$

$$\therefore \lambda^2-\mu\nu=\tfrac{3}{4}(a^2+b^2+c^2)(x^2+y^2+z^2)$$

$$\therefore 4\lambda^2-3(a^2+b^2+c^2)(x^2+y^2+z^2)=4\mu\nu$$

$$\therefore 4\lambda^2-3\lambda(a^2+b^2+c^2)(x^2+y^2+z^2)=4\lambda\mu\nu \quad \ldots\ldots\ldots\ldots(A)$$

Similarly $4\mu^2-3\mu(a^2+b^2+c^2)(x^2+y^2+z^2)=4\lambda\mu\nu \quad \ldots\ldots\ldots\ldots(B)$

and $4\nu^2-3\nu(a^2+b^2+c^2)(x^2+y^2+z^2)=4\lambda\mu\nu \quad \ldots\ldots\ldots\ldots(C)$

Hence the desired relation is

$$2\lambda\mu\nu-(b-c)(c-a)(a-b)(x-y)(y-z)(z-x)=27abcxyz.$$

Now it is at once obvious that when $a=0$ (and so $c=-b$)

$$2\lambda\mu\nu=2\,.\,b(y-z)\,.\,b(z-x)\,.\,b(x-y)=2b^3(y-z)(z-x)(x-y).$$

And an equally obvious expression which will yield the same result when $a=0$ is $(b-c)(c-a)(a-b)(x-y)(y-z)(z-x)$. Hence

$$2\lambda\mu\nu-(b-c)(c-a)(a-b)(x-y)(y-z)(z-x)$$

has factor a.

Similarly it has factors b, c, x, y, and z Thereafter the number 27 is at once obtained. A similar result is found thus:

$$a+b+c=x+y+z=u+v+w=0$$

Let $ax+by+cz=\lambda$, $ay+bz+cx=\mu$, $az+bx+cy=\nu$

$xu+yv+zw=\lambda_1$, $xv+yw+zu=\mu_1$, $xw+yu+zv=\nu_1$

$ua+vb+wc=\lambda_2$, $ub+vc+wa=\mu_2$, $uc+va+wb_2,=\nu_2$.

Then

$$(\lambda^2-\mu\nu)(\lambda_1^2-\mu_1\nu_1)(\lambda_2^2-\mu_2\nu_2)=27\,\Sigma a^2b^2\,.\,\Sigma x^2y^2\,.\,\Sigma u^2v^2.$$

Again, if $a+b+c=0$ and we define thus :

$$\lambda=2(a+\kappa)(b-2\kappa)+(c+\kappa)^2$$
$$\mu=2(b-2\kappa)(c+\kappa)+(a+\kappa)^2$$
$$\nu=2(c+\kappa)(a+\kappa)+(b-2\kappa)^2$$

then $$\lambda^2-\mu\nu=3\{(ab+bc+ca)+3\kappa(b-\kappa)\}^2.$$

If now λ_1, μ_1, ν_1, and λ_2, μ_2, ν_2 are obtained cyclically then

$$(\lambda^2-\mu\nu)(\lambda_1^2-\mu_1\nu_1)(\lambda_2^2-\mu_2\nu_2)=27\{(ab+bc+ca)^3+27\kappa^3(abc-\kappa^3)\}^2$$

35, Spottiswoode Rd., Edinburgh. NEIL Y. WILSON

2701. A five-finger exercise in the scale of ten.

Write down (2×3) different numbers, each containing the digits 7, 8, 9. Find their sum. Then, with the digits of this Exercise in the same order, show your Answer.

(Ans., $2\times37\times8\times9$.)

R. S. W.

2702. Stokes's theorem for n-space.

Consider an infinitesimal parallelogram whose sides are $d\mathbf{r}$, $\delta\mathbf{r}$. We define the *tensor element of area* by

$$d\Sigma=\tfrac{1}{2}(d\mathbf{r}\,;\,\delta\mathbf{r}-\delta\mathbf{r}\,;\,d\mathbf{r})$$

We define similarly the *vorticity tensor* by

$$Z=\tfrac{1}{2}(\nabla\,;\,\mathbf{q}-\mathbf{q}\,;\,\nabla)$$

Since the double scalar product of a symmetric and an antisymmetric tensor is zero, we have to the first order

$$\int d\Sigma\,..\,Z=d\Sigma\,..\,\nabla\,;\,\mathbf{q}$$
$$=\tfrac{1}{2}d\mathbf{r}\,.\,([\delta\mathbf{r}\,.\,\nabla]\mathbf{q})-\tfrac{1}{2}\delta\mathbf{r}\,.\,([d\mathbf{r}\,.\,\nabla]\mathbf{q})$$
$$=\tfrac{1}{2}(d\mathbf{r}\,.\,\delta\mathbf{q}-\delta\mathbf{r}\,.\,d\mathbf{q})$$

Again to the same order

$$\int d\mathbf{r}\,.\,\mathbf{q}=d\mathbf{r}\,.\,(\mathbf{q}-\tfrac{1}{2}\delta\mathbf{q})+\delta\mathbf{r}\,.\,(\mathbf{q}+\tfrac{1}{2}\delta\mathbf{q})$$
$$-d\mathbf{r}(\mathbf{q}+\tfrac{1}{2}\delta\mathbf{q})-\delta\mathbf{r}(\mathbf{q}-\tfrac{1}{2}d\mathbf{q})$$
$$=\delta\mathbf{r}\,.\,d\mathbf{q}-d\mathbf{r}\,.\,\delta\mathbf{q}.$$

Thus

$$\int d\mathbf{r}\,.\,\mathbf{q}=-2\int d\Sigma\,..\,Z$$

or for any contour C and diaphragm S.

$$\int_C d\mathbf{C}\,.\,\mathbf{q}=-\int_S d\Sigma\,..\,[(\nabla\,;\,\mathbf{q})-(\mathbf{q}\,;\,\nabla)]$$

and this is Stokes's theorem. L. M. MILNE-THOMSON

1883. The Nullarbor plain on the east-west line from Port Augusta to Kalgoorlie 297·3 miles of [railway] line absolutely straight except that the earth's curvature is said to make it twelve feet higher at one end than the other.—*The Listener*, March, 26, 1953, p. 515. [Per Mr. A. R. Pargeter.]

PSEUDARIA

12. The conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0 \qquad (a>b)$$

has two real foci $(0, \pm\sqrt{(a^2-b^2)})$, the corresponding directrices are real, and it has real eccentricity. The conic is the locus of points for which the ratio of the distance from a focus to the distance from the corresponding directrix is equal to the eccentricity. Why has the conic no real points?

E. J. F. P.

13. All straight lines are parallel.

For, (i) straight lines may be regarded as circles of infinite radius, (ii) all circles pass through the circular points at infinity, and (iii) straight lines which pass through the same point at infinity are parallel.

RICHARD K. GUY.

14. Rectification.

There is an excellent example in Durell and Robson, *Elementary calculus*, II, p. 329, which reads:

"Find the length of the arc of the curve given by $x=\cos t \sin^2 t$, $y=\sin t(1+\cos^2 t)$ from $t=0$ to $t=\frac{1}{2}\pi$."

It is found that

$$ds=\pm(2\cos^2 t-\sin^2 t)\,dt,$$

and the correct sign must be chosen (the positive sign for $0\leqslant t\leqslant\alpha$, and the negative sign for $\alpha\leqslant t\leqslant\frac{1}{2}\pi$, where $\alpha=\tan^{-1}\sqrt{2}$), a point that is easily overlooked.

Sketching this curve provides a good exercise and the presence of the cusp at $t=\alpha$ explains the change of sign in ds as t passes through the value α. It is perhaps not generally known that this curve is the envelope of the side BC of a rectangle $ABCD$ of unit width AB when A and B slide along two fixed perpendicular lines OX, OY. This knowledge may make the example seem rather less artificial.

Another example is provided by the envelope of the tangent at a given point on a circle which rolls without slipping along a straight line. The envelope, which like the last is most readily found with the help of the concept of the instantaneous centre, has cusps and in more ways than one has a shape which is rather unexpected.

C. M. SEGEDIN.

15. $\tan\frac{1}{2}\pi$ and $\cot 0$ are infinite.

Find analytically the points P on the ellipse $x=a\cos\theta$, $y=b\sin\theta$ for which OP is perpendicular to the tangent at P.

The gradient of OP is $(b/a)\tan\theta$, while the gradient of the tangent is $-(b/a)\cot\theta$, and these are at right angles if $-1=-b^2/a^2$, from which $a=b$. This does not answer the question.

C. M. SEGEDIN.

16. Surface of Revolution.

If the portion of the curve $xy=1$ for $x\geqslant 1$ is revolved about Ox a surface like an infinite hunting horn is produced. The area of this surface is infinite, but the volume enclosed is π cubic units. One comment on this is "How odd, that although an infinite amount of paint would be needed to cover the inside surface it would take only π units to fill it!"

A. P. R.

REVIEWS

Geometry for Sixth Forms. By C. O. Tuckey and F. J. Swan. 2nd Ed. Pp. vi, 248. 10s. 6d. 1955. (Longmans)

The first part of this book consists of a review of " ordinary level " geometry. The treatment is informal, and free use is made of trigonometry. There follow sections on concurrency, collinearity, the circles of a triangle, converses and constructions. A unique feature here is a collection of hard riders " for those who like them hard ". The second part treats of solid geometry. Lines, planes, solid angles, polyhedra, pyramids, tetrahedra, cone and sphere, spherical triangles are introduced. Their main properties are developed and proved. Calculus is used in connection with the mensuration of the cone and sphere.

The third part of the book deals in a more formal way with the modern geometry of the triangle, and of one and two circles. The topics include pole and polar, inverse points, orthogonal circles, radical axis, and the complete quadrilateral. The parabola, ellipse and hyperbola are introduced by pure geometrical methods, and analysis is brought in when it can serve the purpose better. The marriage is a happy one. The last part of the book, entitled " Correspondence of Figures " treats of centres of rotation, direct and inverse similarity, orthogonal and conical projection. Mention is made, and there is some development of Chasles' theorem, Pascal's theorem, inversion, duality and reciprocation. Each part of the book has about 60 pages. Answers to the numerical questions are given and there is a subject index.

Into all parts of the book the authors have put most interesting material, carefully collected from teaching experience. This illuminates parts where the emphasis has been on solid thinking by the pupil, for considerable stress is laid on inductive methods. About one hundred questions are provided, which have been taken from advanced level papers. These are mainly connected with the subject matter of the first two parts of the book, the third part having rather fewer. They are in addition to an otherwise abundant supply of riders.

In the preface, the authors express the hope that a pupil will be able to work through the book with a minimum of guidance. If this hope is to be fulfilled a selection of the sequence to be followed and of the examples to be worked, will be needed. Otherwise the pupil will be held up by work of too great technical difficulty, with consequent loss of momentum and interest. Sometimes too, the lack of a label showing which statement is a definition and which is an enunciation or a conclusion, could be puzzling to a pupil working on his own.

J. W. Hesselgreaves

Topological Transformation Groups. By D. Montgomery and L. Zippin. (Interscience tracts in pure and applied mathematics No. 1). Pp. 281. $5.50. 1955. (Interscience Publishers Inc., New York & London)

Mathematics must by now be the most fragmented of the sciences, and he who would learn about what is being done in fields other than his own may find himself faced with a sentence like this : " Considering quasi-scaffolds which are not d-orthomorphic, we see that . . .". Of course the solution of difficult problems are likely to be difficult reading, but the reader of sentences like that quoted may suspect—and often rightly suspect—that the difficulty lies only in the profusion of jargon and the unnecessary attempt at complete generality, and not at all in the underlying thought. We should then be particularly grateful to the authors of this very fine book ; it describes recent work on difficult problems, it is nearly self-contained, and it carefully avoids useless generality and unnecessary terminology.

First and foremost the book contains an account of the affirmative solution

(by Gleason and the authors) of Hilbert's fifth problem—that is a proof that every locally euclidean topological group is a lie group. (In a lie group local coordinates may be introduced in which the group operations are represented by *analytic* functions and so the techniques of analysis are available for the further study of the group.) Secondly the book describes researches—mostly by the authors—into the transformations of manifolds by continuous groups.

The first two chapters cover, roughly speaking, what was known about locally compact (hereafter " l. c.") topological groups before the last war. They include all the necessary topological and group-theoretic definitions and elementary theorems, the definition of an invariant measure, the solution of the fifth problem for compact and abelian groups, and the theorem of Von Neumann which states that a l.c. group which can be faithfully and continuously represented by a group of matrices is a lie group. The only results which are used without proof concern the eigen-functions of integral equations. Naturally the authors have selected chiefly those results which are required for subsequent chapters, and the reader may sometimes feel a need for more flesh with which to clothe the bones he has been given (and he may satisfy his need by turning to Pontrjagin's classic " Topological Groups "). But what is proved is well proved—the style is neither too condensed nor too discursive.

The next two chapters deal with the fifth problem. It is shown that a l.c. group which does not have arbitrarily small subgroups is a lie group. The first step is to show that the one-parameter subgroups of G form a finite dimensional vector space V. (In the original papers this was done by considering G as a group of operators on the space of square-integrable functions over G; in the book this use of Hilbert space has been confined to two similar and rather difficult lemmas. One wonders whether it might not be possible to prove them also by more direct methods.) The inner automorphisms of G give a continuous representation of G by linear transformations of V, and so by Von Neumann's theorem, there is a factor group G/H which is a lie group. The investigation of H is rather delicate; if G is connected then H is its centre, and hence—being abelian and l.c.—is also a lie group. And it is shown that the extension (in this case G) of an abelian lie group H by a lie group (in this case G/H) is also a lie group.

It now remains to show that a locally euclidean group does not have small subgroups. In the original work this proved a formidable task, but it is done rather simply in the book. It is first shown that any l.c. group G (or rather an open subgroup of G) can be approximated to by lie groups, G being the " projective limit " of a set of such approximations. (This shows that every l.c. group is a " generalised lie group "). And then, using these approximations, it is shown that a finite-dimensional group is locally the direct product of a lie group and a totally disconnected group, from which the final result follows.

The last two chapters concern groups of transformations of manifolds, and are not easy to summarise. They contain some useful technical results concerning differentiability and the existence of cross-sections of orbits, some suggestions of fruitful lines for further research, and some interesting cases where the transformations turn out to be particularly simple. The most striking of these is the proof that a compact connected group of transformations of euclidean three-space must be a group of rotations. These chapters require some knowledge of combinatorial topology.

This is not an easy book, but I hope I have said enough to show that it is a very absorbing one. I have only one criticism: one gets the hang of a subject as much by illustrative examples and counter examples as by formal proofs, and though there are quite a few examples in the book they tend to be collected together in particular sections; there should be more of them, and they should be woven more closely into the commentary on the theorems.

R. O. GANDY

Elements de mathématique. By N. BOURBAKI. Book I. Théorie des ensembles. Chapter III. Ensembles ordonnés. Cardineaux, nombres entiers. Pp. 118. 1,500 Fr. 1956. (Actualités Scientifiques, 1243, Paris)

The first two chapters of Book I introduced the elements of formal logic and a theory of sets of the Zermelo-Fraenkel type. In Chapter III this account of set theory is continued to give a brief survey of transfinite cardinals. This chapter opens with a study of order relations in which the familiar properties of bounds and limits are derived with greater generality than in the customary foundation course in analysis; this generality is a result of studying (partial) ordering relations, relations $R(x, y)$ which have the ordering properties $R(x, y)$ & $R(y, z) \rightarrow R(x, z)$, $R(x, y)$ & $R(y, z) \rightarrow x = y$, $R(x, y) \rightarrow R(x, x)$ & $R(y, y)$ but do not necessarily satisfy the disjunction

$$R(x, y) \text{ or not} - R(x, y)$$

for all values of x, y.

The theory of cardinals presents some points of novelty. A set of cardinal α is defined to be finite if $\alpha \neq \alpha + 1$ and an infinite set is defined negatively as one whose cardinal is not finite. In a book on set theory it is a little surprising to find several pages devoted to properties of the binomial coefficients.

After a very leisurely start the last few sections proceed at a very brisk pace and a good deal of information about Aleph numbers is packed into a few pages, but this is not the book for anyone who wishes to make a detailed study of the modern theory of transfinite numbers. R. L. GOODSTEIN

The Theory of Numbers. By BURTON W. JONES. Pp. xi + 143. 24s. 1956. (Constable, London)

This introduction to elementary number theory covers more ground than is usually devoted to the subject in a school algebra, since it includes nonlinear congruences, the law of quadratic reciprocity and the expression of numbers as sums of two squares. The style is pleasant and easy to read, and proof and explanation are sound, clear and convincing. An ideal book for the mathematical specialist to read in his last year at school (if not much earlier) and for the interested amateur.

The chapter on continued fractions is the longest and most technical in the book, and deals with closeness of approximation to surds and with Pell's equation. Rather surprisingly the connection between continued fractions, modular inverses and the Chinese remainder problem is not mentioned. The familiar general solution of Pell's equation $x^2 - Dy^2 = 1$ by the expansion of $(p + q\sqrt{D})^n$ is proved, although in fact this is a most impractical way of finding numerical solutions. Thinking about this question more than 20 years ago the reviewer discovered the formula

$$x_{n+1} = 2x_n^2 - 1, \quad y_{n+1} = 2x_n y_n$$

giving an infinite succession of solutions when an initial solution x_0, y_0 is known, which is perhaps worth mentioning here.

The opening chapter deals not with number theory as the term is usually understood but with the foundations of arithmetic, and consists in a brief sketch of an axiomatic development of natural numbers, integers, rationals, real and complex numbers. The motivation in the choice of axioms is well brought out, but the treatment is too brief to be quite sound. To give an instance, it is not satisfactory to introduce $-b$ as satisfying the equation $b + -b = 0$ by definition, since $+$ has so far been defined only for natural numbers, and in fact the definition $-a + -b = -(a + b)$ comes on the next page; a correct theory in terms of ordered pairs is however given in the set of examples following this section (where for $(b + b, b)$ in example 9 should stand $(y + b, b)$). R. L. GOODSTEIN

Tables of Weber Parabolic Cylinder Functions. Edited by J. C. P. MILLER. Pp. 233. 63s. 1955. (Her Majesty's Stationery Office, London)

This important volume is sponsored by the National Physical Laboratory through its Mathematics Division. The extensive mathematical introduction which occupies over one-third of the book is due to Dr. Miller, who also had a large share in the supervision of the numerical computations, the great majority of which were carried out by Scientific Computing Service Limited.

The tables relate to solutions of the differential equation

$$y'' + (\tfrac{1}{4}x^2 - a)y = 0,$$

but the mathematical introduction deals also with the cognate equation

$$y'' - (\tfrac{1}{4}x^2 + a)y = 0.$$

Solutions of both equations are required in applications, but solutions of the first equation have been selected for tabulation, since existing tables relating to this equation are of the slenderest, whereas a fair amount of numerical information (though less than one could desire) about solutions of the second equation can be gleaned from existing tables of functions such as Hermite polynomials and functions, the derivatives and integrals of the probability integral, and Dawson's integral.

It should be pointed out that most of the mathematical introduction (namely, all of it except 13 pages devoted to a description of the tables and their preparation) is likely to be of interest to mathematical analysts who have no concern with tables. Sections on the study of differential equations by approximation and on the definition of standard solutions are certainly of general interest. Further sections are devoted to integral forms and to asymptotic expansions and related approximations, as well as to relations with Bessel, confluent hypergeometric, and other functions, and to requisite gamma function formulae. Much of this mathematical development is to be read in conjunction with 29 valuable pages on functions and formulae, and 3 pages of bibliography and references. A number of the formulae worked out are new or derived within the last decade, and the wealth of information is such that the introduction must be considered as a contribution to the literature of mathematical analysis.

The main numerical table is Table I (pages 94 to 177), which tabulates two independent solutions, called $W(a, x)$ and $W(a, -x)$, of the first equation mentioned above. Values are given to about 8 figures for $a = -10(1)+10$ and $x = 0(\cdot 1)10$; there are 4 pages for each of the 21 values of a. Reduced derivatives, $h^n W^{(n)}/n!$, where $h = 0\cdot 1$, are given, up to $n = 9$ in some cases, for use in interpolation with respect to x. Table I was constructed by numerical integration of the differential equation. Tables II and III (pages 178 to 222) give the values of various important auxiliary functions for the same values of a. On the last 11 pages are tabulated various functions of a useful in the present context. Interpolation of Tables I to III with respect to a will frequently be possible to only about 5 figures, owing to the large interval in a; the author suggests that tables at an interval of, say, 0·2 in a would be needed for full interpolability. What is provided in the present volume is, however, an enormous advance on anything previously available, and will suffice for many purposes.

The printing and general format are excellent, being not unlike those of the well-known British Association and Royal Society series of mathematical tables. It seems a pity that such a valuable work has been printed on paper which is somewhat thin; but perhaps we should be thankful that price-inflation has been kept under control. Although the year displayed on the title-page is 1955, publication occurred in 1956. A. FLETCHER

1. Eight and Nine Place Tables of Elliptical Functions. By M. SCHULER and H. GEBELEIN. Pp. xxiv, 296. DM. 58.00. 1955. (Springer, Berlin, Göttingen, Heidelberg)

2. Five Place Tables of Elliptical Functions. By M. SCHULER and H. GEBELEIN. Pp. xi, 114. DM. 29.60. 1955. (Same publishers)

The smaller of these two volumes is, broadly speaking, a condensed version of the larger, adapted to computations with fewer figures, and designed for the use of physicists and engineers. Both volumes tabulate Jacobian elliptic functions and theta functions (or auxiliary functions simply connected with these) with arguments which, as far as bivariate elliptic tables are concerned, are novel. Previous tables have mostly used as the modular argument either the modular angle θ or the squared modulus $k^2 = \sin^2\theta$, and as the other argument either Legendre's amplitude ϕ or the argument u of the Jacobian elliptic functions or the argument $x = \pi u/2K$ of the theta functions. Like others who have used elliptic tables for practical purposes, the senior author (Professor Schuler) has found difficulty in interpolating bivariate tables with θ as the modular argument (he does not mention tables having k^2 in place of θ). He felt that it might be better to use Jacobi's nome q as the modular argument. The junior author sought systematically for functions sufficiently nearly linear or quadratic in both variables to promise reasonably simple interpolation over most of their domain. He decided to take as the second argument $z = \cos 2x = \cos(\pi u/K)$.

In both volumes the elliptic functions are tabulated by means of tables of $\log(\mathrm{sn}\, u/\sin x)$, $\log(\mathrm{cn}\, u/\cos x)$ and $\log \mathrm{dn}\, u$, all logarithms being to the base 10. In the larger volume the arguments are $q = 0(\cdot01)\cdot55$ and $z = -1(\cdot05)+1$; in the smaller, $q = 0(\cdot01)\cdot50$ and $z = -1(\cdot1)+1$.

In the smaller volume, the functions $\overline{G}(q, z)$ and $\overline{H}(q, z)$ tabulated for the determination of the theta functions may be defined by $\vartheta_1(x) = 2q^{1/4}\sin x\, \overline{G}(z)$ and $\vartheta_2(x) = \overline{H}(z)$. The arguments are again $q = 0(\cdot01)\cdot50$ and $z = -1(\cdot1)+1$.

In the larger volume, the greater number of figures makes it necessary to tabulate $G(q^4, z)$ and $H(q^3, z)$, where $\overline{G} = 1 - q^2G$ and $\overline{H} = 1 + qH$. The arguments are $q^4 = 0(\cdot001)\cdot100$ and $q^3 = 0(\cdot002)\cdot176$ respectively, and $z = -1(\cdot05)+1$ in both cases.

The authors state that the advantages of their system would be lost if their bivariate tables were interpolated by ordinary bivariate methods. Interpolation is meant to be performed first in one variable and then in the other, either way round according to circumstances. Consequently all the tables so far mentioned are given twice over, first with q (or q^3 or q^4) as top argument and z as left argument, with differences in z, and then *vice versa*. All these bivariate tables have first differences (in one or other variable) throughout, and second differences when necessary; in the smaller volume, the second differences are modified when necessary, and are then printed in italics.

Both volumes also contain univariate tables relating to modular quantities, such as tables of θ, $\log\sec\theta$, K and K/E with argument q (or q^3 or q^4), and tables of $1/(1-q)$, K and K/E with argument $\log\sec\theta$, part of the latter being derived from a table by E. L. Kaplan. The smaller volume also contains a table of Everett coefficients for interpolating values of a function and of its first derivative.

It is not likely that the new tables will supersede those with θ or k^2 as modular argument, partly because these arguments facilitate work with functions of the complementary modulus, and partly because in some applications θ or k^2 has a simple physical significance, so that the tables are useful for tabular values of θ or k^2, with interpolation in the other variable only. The new tables should rather be considered as extending the facilities available to

computers. A glance at the differences (especially in the smaller volume) will show how successful the authors have been in producing interpolable tables. Anyone confronted with many unavoidable interpolations in both variables is likely to be grateful that the authors have not quailed before the task of constructing tables on a new system.

Occasionally one feels that the detailed arrangement of the tables could be improved. It is a pity that values of θ are given in degrees, minutes and decimals of a minute instead of in degrees and decimals of a degree. A skilled computer might prefer to perform his interpolations with less use of forward differences than in the worked examples. Both volumes have title and text in German and English, but the English translation in the smaller volume suffers badly from misprints.

A. FLETCHER

Index mathematischer Tafelwerke und Tabellen. Index of Mathematical Tables. By K. SCHÜTTE. Pp. 143. DM. 14.50. 1955. (R. Oldenbourg, Munich)

This slim volume gathers together a considerable number of references to tables relating not only to mathematics but also to physics, chemistry, astronomy and astrophysics, geodesy and geophysics, nautical and aeronautical determination of position, meteorology, astronautics, weights and measures, and so on. The preface and headings are given in German and English, the latter sometimes misleading. The references are mainly to books, but a number of papers are also mentioned. Russian titles are given in German translation.

The sixteen chapters are divided into a total of more than 130 subdivisions. While this is not a large number in relation to the field to be covered, so that no great amount of detail can be expected, and while the scheme of classification appears occasionally to be defective, it would be possible within the general scheme of the book to produce a useful work of reference. The reviewer has in fact found the volume useful and interesting, especially for its numerous references to German publications in a subject whose bibliography has hitherto been too largely left in the hands of British and American authors. It must, however, be said that the volume would have been still more useful if it had been more accurate, more specific and more judicious.

Works by J. C. P. Miller, correctly so called on page 107, are attributed to J. P. C. Miller on page 32, L. C. P. Miller on page 87, and F. Miller on pages 6 and 7. On the principle that errors in the early letters of proper names may cause most trouble in consulting alphabetical indexes, it may be mentioned that "Cryde" on page 62 should be Pryde, and "Eversky" on page 31 should be Esersky. These last errors (and other similar ones) recur in the Index of Authors at the end of the book. The information about Pryde's table is wrong; it gives versed sines, not haversines.

References to periodicals usually lack page numbers; sometimes even the volume number is lacking. For instance, on page 31 a paper by P. M. Woodward is said to be in "Philos. mag., J. theor. exper. appl. physics, London 1948". Surely "Phil. Mag., (7) 39, 594–604, 1948" would be more useful, as well as shorter. The title of the paper is incorrectly copied.

Reference to books by the date of the edition or impression seen, without indication of the date of first publication, is generally unsatisfactory, and sometimes highly so. For example, the National Bureau of Standards published 16-decimal tables of natural logarithms of decimal numbers in two volumes, covering the ranges from 0 to 5 and from 5 to 10. Both volumes appeared in 1941. On page 41, however, the first volume is mentioned as of date 1953 ("Re-issue of Mathem. table 10"), and the companion volume is not mentioned. Moreover, this occurs under the heading "Natural antilogarithms

(powers of e^x) " apparently designed for tables of e^x. Of the four entries in this subdivision, one is this half-table of $\log_e x$, and another is a table of 10^x by Filipowski! Other tables of e^x are, however, given on page 70 under the heading " Exponential functions ".

Apart from faulty allocation to subdivisions, the selection of tables sometimes includes trivial items and excludes items of importance. Under " Elliptic functions and integrals " there is no mention of the important tables of 1947 by G. W. and R. M. Spenceley (not, as on page 43, G. W. R. M. Spenceley, nor, as on page 132, G. E. and R. M. Spenceley). The tables of Comrie are by no means adequately included. Modern computations of e and π go far beyond the two mentioned on page 65.

These details have been given merely for the purpose of illustration. A full list of the errors and infelicities in the work would require more space than is appropriate in a review. It is clear that the author's good intentions have gone sadly astray.

A. FLETCHER

1. Lineare Algebra. 2. Normalformen von Matrizen (*ibidem I* 1, 7). By G. PICKERT. (**Enzyklopädie der Mathematischen Wissenschaften, I 1, 6.**) DM. 7.50. 1955. (Teubner, Leipzig)

1. Linear Algebra no longer means Matrices and Determinants, even less Determinants and Matrices. It is rather the theory of finite-dimensional vector spaces or, more generally, of finitely generated modules over a field or a ring. The main topics are linear mappings of one module into another, the existence of a basis, linear and bilinear forms, the dual space and similar concepts presented in an abstract guise tempered by geometrical nomenclature. Matrices appear only in the formal expressions for relations between vector spaces when their elements have been referred to particular bases.

The composition of several modules gives rise to the ideas of a tensor product and of a multilinear form. This leads naturally to the discussion of outer multiplication (Grassmann algebra) of modules. The determinant is then introduced as the function which expresses the isomorphism of the nth outer power of an n-dimensional vector space over a commutative ring with that ring itself. The usual expansion formulae for a determinant can be elegantly derived by this method as also can such special results as Kronecker's identities between the minors of a symmetric matrix.

The spirit pervading the selection and presentation of the subject-matter is strongly influenced by the great work of N. Bourbaki to whom frequent references are made. Since the older literature is well covered by earlier reports, the author has confined himself on the whole to works that have appeared since 1932.

2. In the article on normal forms of matrices the emphasis is laid on the algebra of matrices over a ring or a field rather than on the underlying vector space. A matrix is treated as an object in its own right and not merely as a device to express relations between vector spaces. Whilst this point of view is now less popular, the specialist working in this field will be glad to have at hand this survey of results that have been obtained during the last twenty years or so. The following subtitles indicate the underlying plan : the characteristic polynomial of a matrix, divisibility of matrices, elementary divisors, similarity, equivalence of pairs of matrices, congruence, congruence of pairs of matrices, unitary similarity and normal matrices.

Since the object of these articles is to report to the expert rather then to teach the uninitiated, the author has adopted a concise style aiming at a high degree of generality and completeness. No one will therefore expect to find it easy or enjoyable to read this book from cover to cover. Nevertheless it constitutes a valuable addition to the reference literature in algebra.

W. LEDERMANN

Basic Statistical Concepts. By JOE KENNEDY ADAMS. Pp. xvi, 304. $5.50. 1955. (McGraw Hill)

In his preface, the author explains that his book has two purposes; the first is to explain the basic concepts of statistical inference, the second to develop understanding of the language of mathematical statistics. The development of the subject differs from that ordinarily adopted in elementary textbooks, since the theory of confidence limits is developed in terms of sampling from finite populations in the first three chapters. Chapters 4–7 discuss the parameters of discrete distributions and the characteristics of the hypergeometric, binomial and Poisson distributions. In Chapters 8–11 this discussion is extended to cover continuous distributions, including the normal, χ^2 and t distributions. These are followed by a chapter on bivariate distributions, correlation and regression analysis; a chapter on the F distribution and analysis of variance; and lastly a chapter on non-parametric methods.

Exercises are provided liberally throughout the book, and the solutions to the odd numbered questions are given. Appendices provide an introduction to the calculus, a number of proofs omitted from the text and a useful selection of mathematical and statistical tables.

From one point of view, Prof. Kennedy has achieved the aims set out in the preface, but it is doubtful if his book will appeal to the student with little or no calculus or statistics for whom it has apparently been written. Combinatorial algebra and the use of classes of ordered pairs can seem very difficult to the student whose mathematics includes no calculus, and the statistical examples of the earlier chapters are so trivial, that they provide no indication of the interest to be found in practical statistics. Other students should find this an interesting book, and the author's constant emphasis on the importance of choosing the appropriate mathematical model for any practical problem is very salutary.

FREDA CONWAY

Statistical Calculations for Teachers. By J. E. JAYASURIYA. Pp. 100. 5s. 1955. (Macmillan)

In one hundred small pages, the author explains how to make a great many calculations with examination marks: he gives methods for comparing and combining different sets of marks and measuring the relationship between them. There is no index, but if one were given it would include Chi square test; confidence level; correlation; critical ratios; normal curve; quartiles; regression equation; sampling distribution; standard scores. This is a great deal to attempt in so small a book, but the exposition is clear and the teachers and student teachers for whom this book is written should be able to follow Mr. Jayasuriya's explanations.

Nevertheless anyone who wishes to make use of such a variety of statistical techniques ought to be willing to tackle a much larger book; and in particular they should be interested in the assumptions, both statistical and educational which are implicit in the calculations. The mechanical application of statistical methods is a dangerous practice. It is true that a warning is given in the first chapter that "marks are but crude approximations to the true values we set out to measure when we administer a test", but insufficient help is given concerning the interpretation of the results of calculations. For instance, the reader is shown how to test his correlation coefficient for significance, but statistically significant correlations are not necessarily meaningful.

It should also be pointed out that the methods of calculation described here involve a number of approximations which, though probably adequate for the type of data assumed for these problems, would not necessarily be acceptable in other cases. There is a misprint on page 87; σ_A and σ_B should be interchanged in the formula given on that page, and the calculation adjusted accordingly.

FREDA CONWAY

School Algebra. By G. H. R. Newth and J. S. Smith. Pp. viii, 383 (with answers). 8s. 3d. 1955. (University Tutorial Press)

This new School Algebra provides in its two parts all that is needed for the Ordinary Level course, whether the subject is intended to be examined in accordance with the old syllabus, or one of the newer alternatives. There is a sound introduction by way of numerical substitution and formulae. Exercises have been well chosen and arranged, and care has been taken to use material with which pupils are likely to be familiar. Directed numbers are brought into the course at an early stage, and there are again plenty of exercises on substitution. A very pleasing feature of the book is its sound treatment of graphs, beginning in Ch. XI. The idea of the graphical representation of a function is stressed at the outset, and throughout the course graphical methods are introduced at appropriate points. The development of algebraic ideas follows generally accepted practice and there are no omissions. Part II contains a valuable chapter on " Introduction to the Calculus ". Here there is plenty to interest and occupy the bright pupil, and the not-so-bright will be helped by the clear bookwork, particularly on time-distance-speed problems. The book ends with a large selection of miscellaneous examination questions taken from various examining bodies.

A clear and attractive type has been used, and the layout of examples and explanations is good and easy to follow. The book is well bound. It is to be hoped that the two parts will be made available separately. Whilst it is a good thing for senior pupils to have both parts at hand for widespread revision, it is expensive and wasteful to put them into the hands of juniors whose needs are met by Part I.

The book fulfils all that the mathematics teacher can ask of it—a useful adjunct to his teaching when he is with his pupils in class, and a reliable standby for homework and private study. J. W. Colley

Elementare Differentialgeometrie. By W. Haack. Pp. viii, 239. DM. 22. 1955. (Birkhauser Verlag, Basel and Stuttgart)

The ideas of E. Cartan on exterior products and exterior derivatives of Pfaffian differential forms are becoming increasingly important in various branches of mathematics. The present book contains, as far as the reviewer knows, the first attempt to present these ideas, and to apply them to differential geometry, in such a way that they could be understood by a student who has just completed a conventional course on differential geometry.

In order to carry out this programme, the author first treats the differential geometry of curves and surfaces in 3-dimensional space by the usual vector methods. Vectors are dealt with in Chapter I, curves in Chapter II, and surfaces in Chapters IV–VIII. This treatment is very clear and can confidently be recommended to honours students who are beginning the subject. Chapter III is concerned with a variable set of perpendicular axes whose origin is a variable point of a curve. This paves the way for the treatment in Chapter IV of a variable set of perpendicular axes whose origin is a variable point of a surface.

Cartan's methods are introduced in Chapter IX, and the author shows how most of the results obtained in Chapters IV–VIII can be obtained much more simply by these methods. Finally, in Chapter X, various existence theorems for surfaces are proved. Since most of this work has been done very recently, the author gives a helpful list of references.

Two minor criticisms may be worth making. Firstly, since this is an introductory work, it is a pity that there are no exercises for the student to solve. Secondly, anyone new to the idea of an outer product may be rather uncertain how an outer product, which is defined abstractly, is connected with the ordinary product. For example, the author says, on page 161, that if $\bar{x}(u, v)$

is the position vector of a point on a surface, then $|\bar{x}_u \times \bar{x}_v|\,[du, dv]$, where $[du, dv]$ is an *outer* product, is the element of the surface area as explained in section 19 : there the element of surface area was shown to be $|\bar{x}_u \times \bar{x}_v|\,du\,dv$, where $du\,dv$ is an *ordinary* product. Now, although this can be justified, it is not obvious, and a little more explanation at this stage would have been helpful.

Nevertheless, this is a fine book, which should help to enable Cartan's ideas to become more widely known. It is beautifully produced, and although the price is rather high by British standards, the book is well worth it.

E. J. F. PRIMROSE

Vector Analysis. By HOMER E. NEWELL, Jr. Pp. 216. 41s. 6d. 1955. (McGraw Hill, New York)

This book on Vectors is divided into two parts the first of which is devoted to general theory and the second to applications. The avowed purpose is to develop the algebra and calculus of Vectors in the way in which the physicist and engineer will want to use them and the principal criterion employed by the author is that the text should be understandable and useful to the serious student. The first half of Part I gives more attention to rigour than is usual in books designed for physicists and engineers, but even where the argument is not entirely rigorous it is usually clearly stated that this is so and the deficiencies in the reasoning are suggested. For this reason Part I of the book should be a valuable text for the better student. There is an admirable selection of exercises. Since the answers are given, the exercises of chapter 7, which are concerned with the fundamental formulae related to the various systems of orthogonal curvilinear co-ordinates in common use, will be found exceedingly useful for reference purposes.

The applications of Part II are restricted to kinematics and a sketch of Electromagnetic theory. The latter although fairly comprehensive is perhaps too concise for the novice, who will find the going difficult. It should however be valuable to a student already familiar with the physical background. This section begins by assuming the validity of Maxwell's equations in the form

$$\gamma\,\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \gamma\,\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \beta \mathbf{J}$$

in which it is assumed that a systematic set of units, determined by γ, β, and by ϵ_0, μ_0 which are later introduced, is employed. To keep the choice of units open these quantities are carried along in the arguments which follow and the various common systems of units are then tabulated at the end of the chapter. It is curious however, that the very next formula quoted, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, requires modification for two of the systems of units mentioned.

Taken as a whole this is a satisfying volume but one or two details did not appeal to the reviewer. The short section 1.6 on groups is in no way essential to the structure of the book and could well have been omitted. In the definition of a differential (p. 36) the statement "Δf is given approximately by df" is irrelevant and in fact dubious unless the nature of Δx is further qualified. The author defines the operator ∇ by the relation

$$\nabla = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\substack{\text{bdy} \\ \text{of } \Delta V}} d\mathbf{S}.$$

This will certainly puzzle many readers who are asked to prove on the next page but one that $\oint d\,\mathbf{S} = 0$. Having insisted on a definition of ∇ which is inde-

pendent of a co-ordinate system, the operator $\mathbf{A} \cdot \nabla$ is defined to be

$$A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}.$$

This approach seems inconsistent, especially since the presentation suggests that the formulae for $\nabla \times (\mathbf{A} \times \mathbf{B})$ and $\nabla(\mathbf{A} \cdot \mathbf{B})$ are valid only in the case of rectangular co-ordinates.

On p. 116 we find the following consecutive sentences " The points at which $\mathbf{b} \neq 0$ are called the vortices of $\mathbf{A}$. Suppose the vortices of $\mathbf{A}$ have continuous first derivatives. . . .". Some readers will know what this means, but others will wonder how a point can have derivatives and others who may have heard of rectilinear vortices may wonder whether there are rectilinear points. It would have been preferable to adhere to the usual practice of distinguishing between a vortex and vorticity.

The paper and printing are excellent. No misprints were detected.

D. E. RUTHERFORD

Matrizenrechnung. By W. GRÖBNER. Pp. 249. DM. 23. 1956. (Oldenbourg, Munich)

The reader of Professor Gröbner's *Matrizenrechnung* need possess no previous knowledge of linear algebra ; but if he is to follow critically the account offered to him by the author, he should have some experience of abstract reasoning and at least a nodding acquaintance with algebraic structures such as those of a field or a principal ideal ring. The book, though moderately short and fairly restricted in scope, is therefore not altogether suitable for beginners. However, the emphasis throughout is on *linear* algebra, and general concepts are only used sparingly, so that the reader who wishes to disregard them and to think solely in terms of matrices with real, complex, or integral elements can usually do so without difficulty. At the same time he will need to exercise some care in discriminating between results involving the real and the complex fields ; in this respect a more explicit formulation of several theorems would have been an advantage.

The first chapter of the book deals with the most elementary properties of vectors and matrices. Vectors are initially defined as ordered sets of n numbers, but the axiomatic treatment follows a few pages later. Chapter 2 is devoted to a somewhat unusual development of determinant theory, based on Grassmann's idea of an outer product of vectors. Chapter 3 is concerned with the theory of linear equations, and Chapter 4 takes up again the discussion of determinants ; it also contains a useful section on compound matrices, a topic normally ignored in elementary text-books. This is followed, in Chapter 5, by a discussion of rational functions of a square matrix and, in particular, of the characteristic equation, the minimum polynomial, and the principal idempotent elements of a matrix. The centre of gravity of the book is reached in Chapter 6 which deals with unimodular equivalence and similarity transformations. Here the theory of elementary divisors is first developed and the criterion for the similarity of two matrices is derived. Then a concise and elegant exposition of the classical canonical form is given, and this is followed by the resolution of the vector space into a direct sum of subspaces invariant with respect to a given linear operator. The final chapter is concerned with congruence transformations and deals with such topics as reduction to principal axes, the law of inertia, and automorphic transformations. A series of problems illustrating and expanding the preceding theory is appended to each chapter, and solutions are briefly indicated in the majority of cases. The inclusions of problems is a welcome feature, especially as most authors of German text-books disdain to make this concession to the reader.

This short summary of contents does less than justice to the distinctive flavour of the book. Professor Gröbner has a light touch, and it is not the least of his merits that he is able to combine strictness of reasoning with a consistently interesting presentation of his material. Altogether, we are indebted to him for a brisk and most readable account of elementary matrix theory.

L. MIRSKY

Integral Functions. By M. L. CARTWRIGHT. Pp. 135. 18s. (Cambridge University Press)

This tract deals mainly with functions $f(z)$ regular and of finite nonzero order ρ in a domain D which may be either an angle or the open plane. In this field the author has done much of the basic work.

A fundamental part is played by the Lindelöf function

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{V(r)}$$

where $V(r)$ is a suitable comparison function. This is related to the maximum and minimum modulus of $f(z)$, lines of Julia (but not of Borel), and in the last chapter a valuable account of the Borel polygon and support sets and functions, for functions of exponential type.

Another highlight is the construction of a Lindelöf proximate order $\rho(r)$ that is analytic in an angle about the positive real axis, whenever $f(z)$ is an integral function of finite nonzero order. By taking $V(r)=r^{\rho(r)}$ and considering $f(z)\,e^{-V(z)}$ the author is led to a simple and elegant treatment of several classical problems without being restricted to functions of mean type.

These topics are not accessible in any other book, certainly not in this price range. The author has included enough basic theory to make her account self-contained. The style is clear and everything is well motivated so that the reader knows what is going on. Thus the book should have a wide appeal to analysts. Much of it will also be new to specialists but the size of the book and attention to clarity and completeness have made it impossible to be exhaustive. The methods and ideas of Nevanlinna are barely used (defect is not defined) and extensions to subharmonic functions are only hinted at. There is however a good bibliography of recent work in the field which will make it easy for any interested reader to study further.

W. K. HAYMAN

Neuere Untersuchungen über eindeutige analytische Funktionen. By H. WITTICH. Pp. 163. DM 25.60. 1955. (Springer, Berlin)

This book is mainly concerned with functions regular or meromorphic in a disk $|z|<R\leqslant\infty$. The basic material consists of a good account of the Wiman-Valiron theory of the maximal term of a power series in Chapter I and a short account of the two fundamental theorems of Nevanlinna but without consideration of the error term $S(r)$ in Chapter II. The author then proceeds to cover the progress in the theory since Nevanlinna's *Eindeutige Analytische Funktionen* appeared in 1936.

Chapters III and V contain results achieved by analytic methods. The latter chapter gives a very fine account of the application of the basic theory to the solution of differential equations. In the rest of the book the motivation is largely geometrical. There is a detailed discussion of the class F_q of simply connected Riemann surfaces over the sphere, all of whose branch points lie over a finite number $d_1, d_2, \ldots d_q$ of base points. The mapping functions onto these and related surfaces have proved very fruitful in the hands of Ullrich and his pupils in constructing examples for the second fundamental theorem and these are treated.

Thus we find examples of functions with an infinite number of distinct

values with positive Verzweigungsindices, which may be irrational, and also examples of irrational defects. Various criteria for the parabolic or hyperbolic type of a surface of class F_q are also given. Partly as an aid to such studies the theory of quasiconformal maps of doubly connected domains is treated in Chapter VI.

It will be obvious by now that this book is essential to any student of the Nevanlinna theory. Here he will find what problems have been solved since 1936 and by whom. To do this he will however have to work. The matter of the book flows on with but few landmarks. On the last page, but hardly anywhere else, we find a theorem in Italics. There are no headings beyond the chapter headings. Proofs are put in if they are short and otherwise the reader is referred to original papers. The bibliography is fairly comprehensive (On page 85 reference should have been made to M. Schiffer, *Q. J. of Math.* (17) (1946) pp. 197–213, which appeared 3 years before the author's work on the subject). A serious student will learn much from this book.

W. K. HAYMAN

Plane Waves and Spherical Means Applied to Partial Differential Equations. By FRITZ JOHN. Pp. ix + 172. 32s. 1955. (Interscience)

This book, the second of a series of monographs edited by L. Bers, R. Courant and J. J. Stoker, is the first connected account of certain methods in the theory of partial differential equations which have so far been available only in papers scattered in the literature. The author develops in some detail the applications of two relatively elementary identities. The first of these expresses an arbitrary function of any number of variables in terms of the spherical means of integrals of the function over hyperplanes, to which the iterated Laplacian is applied. It is due to J. Radon and is here called the Radon transform. The second identity, due to Prof. John himself, expresses an arbitrary function in terms of its integrals over spheres whose radii are bounded away from zero.

The Radon transform is in effect a decomposition into plane wave functions that is to say functions of the form $g(y \cdot x)$ where $y \cdot x$ is the scalar product of two vectors one of which is fixed. (The Fourier transform is a decomposition of this type.) The advantages of the Radon transform, at least in some problems, are brought out in striking fashion by a concise and elegant treatment of the initial value problem for a homogeneous hyperbolic equation with constant coefficients. Another important application which is set out in detail is the construction of a fundamental (elementary) solution of a linear elliptic equation of any order with analytic coefficients; the method is also applied to a linear elliptic system. The elementary solutions constructed are in general only valid in the small, but for equations with constant coefficients explicit solutions involving only quadratures are obtained.

The second identity already referred to is demonstrated in the course of a discussion of the properties of the iterated spherical means of a function. Its main application is to the differentiability properties of the solutions of both linear and non-linear elliptic equations or systems of such equations; the method is simple and transparent. It is also applied to the discussion of the relation between weak and strict solutions of linear elliptic equations. The author is able to extend some of his results on differentiability to non-elliptic equations by considering the integrals of solutions along time-like curves.

Other topics treated in this book are the identity of Asgeirsson and a more general identity due to Aughtum S. Howard and the problem of determining a function from its integrals over spheres of unit radius. This problem is related to the "functions periodic in the mean" introduced by Delsarte; its solution is obtained both in terms of the Radon transform and by the combination of suitable initial value problems of the wave equation.

As this brief summary shows, the book covers a good deal of ground in spite of its small size. It is nevertheless very readable, as a result of the author's clear and elegant exposition. It deals with the theory of partial differential equations as a branch of analysis and there are no references to physical applications. It should however prove of some value to applied mathematicians as well; the Radon transform in particular may well prove a useful tool in a variety of problems.

F. G. FRIEDLANDER

Eléments de Mathématique, XVIII and XIX. Livre V, Espaces Vectoriels Topologiques. Chaps. III–V, Pp. 189. 2,000 fr. Fascicule de Résultats. Pp. 39. 400 fr. 1955. (Hermann & Cie., Paris, Actualités Sc., Nos. 1229 and 1330)

The chapters of this book combine with unusual fortune the functions of forming a necessary part in an encyclopedic whole, and of filling a gap in mathematical literature which has been sorely felt for many years. Since the book by Banach there has been no general work on the theory of linear spaces capable of taking its place and of bringing it up to date, though there have been some excellent books on particular aspects of the subject. Quite largely owing to the work of the school of Bourbaki, it has been shown that the theory of Banach spaces is not wide enough for all the applications of linear spaces which are needed in analysis, and that an extension of the theory, to a class of spaces wide enough to include for example the spaces on which Laurent Schwartz' distributions are defined, but not so wide that the major theorems of Banach space theory lose their validity, was possible and essential.

Chapter III of this book deals with spaces in which the Banach-Steinhaus theorem, and the important closed graph theorem of Banach, are valid. Chapter IV discusses the dual space of a topological vector space, with the topologies defined on such a space in terms of its dual, and with the special results which hold for Frechét and Banach Spaces. Chapter V deals with the elementary—that is, the geometrical—parts of the theory of Hilbert Space. The book ends with an excellent historical note, and with a Dictionary giving translations into the Bourbaki terminology of terms used by other writers in French, English and German. References to the location of definitions of important terms are given.

The Fascicule de Résultats is a lucid resumé of the theory developed in Livre V.

The work is one which will prove indispensable to all analysts interested in the theory of linear spaces. A necessary prerequisite for the reader is knowledge of the matter in the first two chapters of Bourbaki's *Topologie Generale* and in the sections dealing with the Baire Category Theorem.

J. L. B. COOPER

Harmonic Analysis and the Theory of Probability. By S. BOCHNER. Pp. 176. 35s. 1955. (California University Press and Cambridge University Press)

The author has in this monograph given a study of the applications of the Fourier Integral, in a very general sense, to Harmonic Analysis and to the study of stochastic processes. The unifying notion of the book is that of the characteristic function of a measure. In the first four chapters, which are a development of the author's earlier work on Fourier Integrals, the main interest is in the discussion of functions which are represented as Fourier transformations of measures, and also of the Laplace transforms; a number of interesting relations with the theory of diffusion processes, of summation of integrals and of partial differential equations are brought out. The fourth chapter begins the study of stochastic processes, and the last two chapters are devoted almost entirely to that subject. As is usual in modern mathematical treatments, the point of view is that which regards probability theory as a branch of measure

theory. A stochastic process is a family of measure spaces, depending on a parameter. The theory developed here goes beyond the normal level of generality in that it allows for quite arbitrary families of measure spaces, and does not confine the set of parameters to being one-dimensional but allows of any directed set as the family of parameters. As a result, a considerable amount of technical apparatus is needed for setting up the definitions. The theory contains much that is of mathematical interest, and no doubt much that is of interest to the student of the more abstract aspects of mathematical statistics : in particular, the theory of what Bochner calls the characteristic functional is developed in the last two chapters ; this reduces in the case of a single probability distribution in an Euclidean space to the ordinary characteristic function, while in the case of a family of probability distributions on Euclidean spaces it enables the entire story of the family to be summarized by an ingenious process.

Written as a monograph, this book is intended for specialists. This is least true of the first three chapters, which contain much of interest to any student of analysis, and which would not require much expansion to make an excellent introduction to the theory of Fourier integrals. The last two chapters are made harder going by the lack of direct indication of the relationship of the problems discussed to more concrete problems of probability theory, and by the great generality of the techniques. The book is well produced, and ends with a very useful series of bibliographical notes. J. L. B. COOPER

Kreis and Kugel. By W. BLASCHKE. 2nd edition. Pp. viii × 167. DM. 18.60 1956. (De Gruyter, Berlin)

The first edition of this book was published forty years ago and a reprint of it was reviewed by T. J. Willmore in the *Gazette* (xxxvi, p. 67). The present edition has been brought up to date by the inclusion of recent results in the theory of convex bodies. R. L. GOODSTEIN

Gesammelte Mathematische Abhandlungen. LUDWIG SCHLÄFLI. Part III. Pp. 402. Fr./DM. 59.30. 1956. (Birkhauser, Basel)

This magnificently produced third volume of Schläfli's works contains papers written between 1866 and 1885 on function-theoretic topics, Bessel functions, elliptic modular functions and Abelian integrals, Legendre functions and Spherical Harmonics. R. L. GOODSTEIN

Elementary differential equations. By W. T. MARTIN and E. REISSNER. Pp. vii + 260. $5.50. 1956. (Addison-Wesley, Massachusetts)

This course on differential equations is planned to interest students reading for a general science degree with an engineering bias. The emphasis is on solving equations and purely theoretical discussion is limited to a few pages in Chapter VI on the existence of a solution of a first order equation.

The book has many attractive features ; it is very well printed with an "open face" page very easy to read ; there are a great many worked examples and a large collection of exercises with answers ; the section on Laplace transforms is admirably helpful and lucid and brings out the virtue of the transform solution of a differential equation very clearly, though unfortunately there is no hint of the inversion problem for the transform.

The most noticeable omission is all reference to operational methods. Nothing is said about the equation $L(D)y = R$ from the operational standpoint, even in the particular cases $R = e^{px}$ and $R = x^n$. Here and there the text suffers from a want of elegance which is particularly noticeable in the sections on simultaneous systems of equations and the determination of the error in Simpsons's rule but the general impression which the book gives is one of practical competence. R. L. GOODSTEIN

Praktische Mathematik. By H. VON SANDEN. 4th Edition. Pp. 155. DM. 7.60. 1956. (B. G. Teubner, Stuttgart)

This is a further enlargement of the third edition published three years ago (*Gazette*, xxxviii, 325). The text is basically the same with five major insertions which make this edition about twenty per cent. longer. About half the new matter deals with von Graeffe's method of solving equations, including the case of complex roots. The other additions deal with interpolation for unequal intervals of the argument, a least-squares method of differentiation. There is also a new brief chapter on Nomography which is adequate within its limits.

J. C. W. DE LA BERE

General Topology. By JOHN L. KELLEY. Pp. xiv, 298. 60s. 1955. (Macmillan)

" This book is a systematic exposition of the part of general topology which has proven useful in several branches of mathematics. It is especially intended as background for modern analysis, and I have, with difficulty, been prevented by my friends from labeling it : What Every Young Analyst Should Know." The friends were surely right ; for while the book is clearly of the greatest value as a work of reference, the main question in the reviewer's mind was to decide to what reader it should be recommended. The contents are somewhat coextensive with Bourbaki's Topologie ; the principal additional features of the present work are (i) its copious collection of highly illustrative example-problems, taken largely from analysis ; (ii) the chapter on Moore-Smith convergence which may well appeal more to analysts than the equivalent theory of filters (incidentally, the author is deceptively mysterious about this equivalence in remark L(f), p. 83, and avoids even mentioning that a uniformity is a filter, p. 176) ; and (iii) the inclusion of a preliminary chapter and an appendix on set theory, the former being more intuitive and the latter precise.

The author admits the difficulty of combining in one book the function of reference and text ; but, though the book is based on " various lectures " by the author and is logically self-contained, the abstractness of the formulations and the generality of the statements of many of the theorems (the author is at pains to avoid point separation axioms) would certainly trouble an audience of British mathematicians quite inexperienced in topology. It would seem then that the book is suitable for the analyst or for the student of topology who has had some grounding in such a work as Kuratowski's Topologie (Vol. I) or Newman's Topology of Plane Sets. Indeed, for such a reader, the book can be highly recommended for its clarity and the vividness of the presentation. These are enhanced by the author's style which, though it might be described as " racy ", appealed strongly to the reviewer. The inclusion of the chapters on set theory seemed an excellent idea and it was very sensible to reserve the rigorous treatment to the end. However, part of the value of Chapter O was diminished by the use of formulations whose significance depended on definitions (e.g., of set, universe) in the Appendix. Thus the definition in Chapter O, " $A = B$ iff for each x it is true that $x \in B$ whenever $x \in A$ ", appears to compare unfavourably with " $A = B$ iff $x \in B$ whenever $x \in A$ ". Again the notion of absolute complement is curious in topology—we are not accustomed to regarding elephants as belonging to the complement of a space. Perhaps also the bracketing convention implicit in, e.g., $A \cap X \sim B$ (p. 3) should have been made explicit or not adopted.

Halmos' conventions of using " iff " for " if and only if " and of putting the symbol ∎ at the end of a proof are adopted and, indeed, in only a few cases were the author's notations anything but extremely welcome. However, the use of the description " indiscrete " for a space X whose only open sets are X

itself and the empty set was regrettable and the use of the term "r-sphere" for the set of points whose distance from a fixed point was less than r conflicted violently with its use by algebraic topologists. The reviewer would also have favoured the Bourbaki convention of writing "increasing" for "non-decreasing", etc.

Two comments, each on a question of individual taste, suggested themselves on reading the book. First, one did not seem to have penetrated very far into active point set topology despite the amount of material presented, and, second, the proofs were sometimes disappointingly attenuated after a liberal supply of preparatory chat. But without doubt very much of this chat was highly illuminating.

The publishers of the book have played one devastating trick on the author: an excellent reference system to previous theorems and results is vitiated by the extreme difficulty of discovering the number of the chapter—it appears once only at the beginning of the chapter. The printing is generally clear and satisfactory but, while admitting that fractions are awkward to print, the use of $2m2^{-n}$ for $2m/2^n$ needs some justifying. The proof reading is imperfect and the reviewer noticed the following misprints.

p. 29, l. 9	for	E_O	read	E_0
p. 38, l. 2	for	indiscrete	read	discrete
l. 3	for	discrete	read	indiscrete
p. 66, l. 2	for	$m \epsilon E$	read	$m \epsilon D$
p. 106, l. 18	for	$X \cdot Y$	read	$X \cdot U$
p. 107, l. 17	for	number	read	member
p. 130, l. 13	for	showed	read	shown
p. 131, l. 24	for	$y/1+x$	read	$y/1+y$
p. 143, l. 19	for	4.6	read	5.6
p. 176, l. 20	for	for $u \epsilon V[x]$ and $v \epsilon V[y]$	read	$u \epsilon V[x]$ and $v \epsilon V[y]$ for
p. 185, l. 21	for	$U_n \sim U_{n-1}$	read	$U_n \sim U_{n+1}$
(and surely we need $d(x, y) < 2^{-n+1}$ in condition (b) and l. 28?)				
p. 198, l. 11	for	$(f(x), f(x))$	read	$(f(x), f(y))$
p. 219, l. 23	for	(F, T) is a compact Hausdorff space	read	(F, T) is a compact space and Y is Hausdorff
p. 221, l. 18	for	$F \times X$	read	$F \times X \to Y$

But the last word should be that this book made most enjoyable reading.

P. J. HILTON

Höhere Mathematik für Mathematiker, Physiker, Ingenieure. By R. ROTHE. Vol. VII. Prepared by W. SCHMEIDLER. Pp. 218. DM 19.80; bound, DM 22.40. 1956. (Teubner, Stuttgart)

Thorough revision of Rothe's book (of which, for example, Volume I is now in its 9th edition) has involved some extension, of which the latest volume is an instance, in order to bring recent developments in mathematics to the service of the engineer and technician. In 200 pages Dr. Schmeidler copes with harmonic functions in two and three dimensions, linear algebra and analysis, and the calculus of variations. These headings are liberally interpreted; for example, integral equations are generously treated, with some indication of numerical methods such as Nystrom's, and a section on the Fourier and Laplace transforms. This close packing has been made possible partly by reducing the wealth of illustrative matter which made the earlier volumes so attractive and partly by very careful attention to concise exposition. With all the care in the world, however, the elliptic functions can scarcely be made attractive in six pages. Thus the book may well serve as a lucid sketch of a number of exceedingly important topics, and a sign-post pointing to the more comprehensive works listed in the bibliography.

T. A. A. BROADBENT

Mathematical Interpretation of Formal Systems. By TH. SKOLEM. G. HASENJAEGER. G. KREISEL. A. ROBINSON. HAO WANG. L. HENKIN. J. ŁOŚ. Pp. viii + 113. 24s. 1955. (North Holland Publishing Company, Amsterdam)

Apart from the addition of an article by Hasenjaeger this book contains the lectures given at the Symposium on the interpretation of formal systems which followed the International Congress of Mathematicians held in Amsterdam in September, 1954.

In the opening lecture Skolem describes the great discovery he made in 1933 of (what is now called) a nonstandard model of arithmetic. Such a model is an ordered set of objects, of ordinal greater than that of the natural numbers, which yet have all the properties of numbers and satisfy the axioms for natural numbers in a formalised arithmetic. Skolem's construction is not finitist in character, and it does not seem possible to find a purely finitist construction in the general case, but Skolem shows that it may be achieved for certain fragments of recursive arithmetic.

Hasenjaeger's paper on definability and derivability is concerned to show that the class of universal consequences of a system with essentially free variables, but with no extra-logical constants, is not axiomatizable.

Kreisel discusses models, translations and interpretations and gives many helpful clues to the part played by these fundamental notions, including a sketch of his "no-counter-example" interpretation which illuminates Brouwer's intuitionistic logic. The paper is unfortunately conspicuous for its failure to appreciate uses of the term "finitist" other than the author's own, a failure which was not apparent in the lecture itself.

Robinson's lecture on ordered structures is again concerned to show the service which mathematical logic can render to Algebra, as is Henkin's lecture on Cylindrical Algebras and Łoś' on *Les classes definissables d'Algebres.*

Wang's lecture on denumerable bases of formal systems discusses the paradoxes of set theory and in particular the Skolem paradox that a consistent formal set theory has a denumerable model even though one of the theorems of the theory asserts that the elements of the theory are non-denumerable. This paradox obliges one to accept the position that every formalised consistent set theory is incomplete in the sense that there must be functions which cannot be expressed in the system. R. L. GOODSTEIN

The Number System. By H. A. THURSTON. Pp. vi × 134. 15s. 1956. (Blackie)

In a recent notice of a reprint of Landau's *Grundlagen der Analysis* Professor Broadbent said that Landau had done the job of setting up the number-system once and for all; Dr. Thurston has shown that the job was worth doing again, and very well it has been done. In the opening chapters the whole programme of the book is carried through informally, bringing out the motivation underlying the formal treatment which follows. A considerable economy is achieved by a free use of the concepts of modern algebra; each new extension of the number system (short of real numbers) is introduced as a dyad of a hemigroup, and all the arithmetical properties are ready to hand from the chapter on groups and hemigroups.

The theory of real numbers is based, not on Dedekind sections but on Cauchy sequences and an attractively presented detailed development is carried out, up to the proof of the exponential laws for a real index.

Thurston reproduces (without acknowledgement) Kalmar's proof, as it is given in Landau's *Grundlagen der Analysis*, that there is unique function $f(x, y)$ satisfying the recursive equations for addition

$$f(x, 0) = x, \quad f(x, y') = \{f(x, y)\}' \qquad (1)$$

(where x' denotes the successor of x). The need for such an existence proof in a book of this kind (which contains no formal logic) is questionable, and the proof itself is confused. We may for instance argue (as elsewhere in the book) that the class of values of y for which the value of $f(x, y)$ is uniquely determined contains all natural numbers since $f(x, 0)$ is given and $f(x, y')$ is uniquely determined when $f(x, y)$ is determined. As to the proof itself Thurston seeks to prove, not that there is an f such that equations (1) hold for all x and y but rather that for each value of x there is a function which satisfies (1) for all y; this is of course an equivalent result but the reader can only be misled by a proof which introduces a function $\theta(x, y)$ for a given x as satisfying (1) for all y and then *defines* the function for the next value of x, namely x', to be $\theta(x', y) = (\theta(x, y))'$, since we have apparently introduced a *second recursive definition*. In fact it is a function of y which we introduce for each value of x, as follows. To the value $x = 0$ corresponds the function $f(y) = y$; if the function which corresponds to a given x is $g(y)$ then the function which corresponds to the next x, namely x', is $h(y) = \{g(y)\}'$, for $g(0) = x$ and $g(y)' = \{g(y)\}'$ so that $h(0) = x'$ and $h(y') = \{g(y')\}' = \{h(y)\}'$. Moreover, in view of the form which the existence proof takes it is not appropriate to prove that there is only one f which satisfies (1) for all x and y, but rather that if there are two sequences $f_n(y)$, $g_n(y)$ which, for each n, satisfy $f_n(0) = n$, $f_n(y') = \{f_n(y)\}'$; $g_n(0) = n$, $g_n(y') = \{g_n(y)\}'$ for all y then $f_n(y) = g_n(y)$. However, since Landau's book was written, Gödel has shown how we may given an *explicit* definition of recursive functions which is certainly preferable to an existence proof. For addition this definition is: $x + y$ is the unique value z for which there is an f such that

$$f(0) = x \quad \text{and, for all } n, \quad f(n') = \{f(n)\}', \quad \text{and} \quad f(y) = z \tag{2}$$

The definition presupposes a proof that for given x, y there is a unique z which satisfies (2); suppose in fact that f, z; g, w both satisfy (2) then $f(0) = x = g(0)$ and if for some n, $f(n) = g(n)$ then $f(n') = \{f(n)\}' = \{g(n)\}' = g(n')$ so that $f(n') = g(n')$, whence it follows that $f(n) = g(n)$ for any n, and therefore $z = f(y) = g(y) = w$. It remains to prove that, for any x, y there is a z and an f such that (2) holds. We start by proving that for any x, there is an f such that

$$f(0) = x \quad \text{and, for all } n, \quad f(n') = \{f(n)\}'. \tag{3}$$

For $x = 0$, the identity function $f(n)$ satisfies the condition (3); and if for some x, $g(n)$, satisfies (3), then for x' we may satisfy (3) by taking $f(n) = \{g(n)\}'$. Let M be the class of values of y for which (2) holds for any x. By (3), 0 belongs to M (with $z = x$); and if y belongs to M, so that there exists f and z such that (2) holds, then since

$$f(0) = x \quad \text{and, for all } n, \quad f(n') = \{f(n)\}', \quad \text{and} \quad f(y') = z'$$

it follows that y' belongs to M, and therefore all y belong to M, which completes the proof.

R. L. GOODSTEIN

Algebra, II. By B. L. VAN DER WAERDEN. 3rd Edition. Pp. viii + 224. DM. 29.60. 1955. (Springer, Berlin)

Apart from a change in paragraph numbering and the corrections of misprints this edition is the same as the second. The first edition was reviewed by W. L. Ferrar in the *Gazette* twenty-five years ago (xvi, p. 147). In the second edition the last two chapters on hyper complex numbers and their representation was considerably expanded. Recently an English translation was made from the second edition of this exceedingly important book.

R. L. GOODSTEIN

Comprehensive Mathematics. By J. D. HODSON. Pp. 320. 8s. (with answers 8s. 6d.) 1955. (Macmillan)

The author is Senior Mathematical Master at a Grammar School, and the scheme used as a basis for the book has been tried out in his department, for both good and weak pupils preparing for Ordinary Level examinations of the General Certificate of Education. The subject matter is arranged so that there is some Geometry, some Algebra, some Arithmetic and some Trigonometry in most of the chapters. For those who like this system the book should be useful, for there is as much correlation as possible, although it must be admitted that there are plenty of topics in mathematics which by their very nature are isolated, and in such a book must therefore appear in the company of unconnected parts of the subject in one chapter.

There are some valuable test papers on the lines of the Alternative Syllabus, and the examples in general are well chosen, notably the problems on income tax and home accounts, navigation, latitude and longitude, plans and elevations.

The final three chapters are on Calculus and Coordinate Geometry, and are intended for the brighter pupils.

There are one or two blemishes. In the definitions of trigonometrical ratios on page 13 the angle X is used, while in the relevant diagrams on page 12, the angles are marked x. In example 5 on page 50, in which lengths are given as x inches, a certain area is to be proved to be $(5/9)x^2$, and there is a similar omission of units in a worked example on page 134. It is unfortunate that in a statement on page 18 about an isosceles triangle we read " each of its angles are 60° ", and again on page 39 that " two sides and a non-included angle is sometimes sufficient ". Many mathematicians will not approve of the terms " horizontal " and " vertical " for the axes in a distance-time graph discussed on page 181.

The book has, however, much to commend it, and pupils who have used it should find the table of contents a useful reference for revision.

J. K. DUDLEY

THE MATHEMATICAL ASSOCIATION

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THE LIVERPOOL MATHEMATICAL SOCIETY

Report for the Session 1955–1956

17th October, 1955. Mr. Cox of the University of Sheffield spoke to the Society on "Astronomy and the Young". Although there is much interest shown by children, the speaker advocated the exclusion of Astronomy from a school syllabus. Nevertheless, many examples can be introduced to enliven the teaching of other subjects, and a number of such illustrations were given, In a School Astronomical Society the aim should be to teach children to relate observations made in the sky with the Copernican model of the solar system.

14th November, 1955. Dr. Bonsall of Newcastle chose for his title "The Mathematical Theory of Games". The theory is concerned with contests, subject to certain rules, between two or more persons, and is complete only for two-person zero-sum games. Using the notion of strategy each such game can be represented by the pay-off matrix. If a saddle-point exists in this matrix, there are definite optimum strategies available to a player intent on winning.

5th December, 1955. To mark the year of the death of Albert Einstein, Dr. Bonnor was invited to address the Society on "The Greatness of Albert Einstein". Much of the theory of special relativity received contributions from Poincaré and Lorentz, whereas the theory of general relativity is entirely due to Einstein. After giving a short description of general relativity and later developments, Dr. Bonnor concluded with an account of Einstein's social and political views.

23rd January, 1956. This meeting took the form of an "Exhibition of mathematical models and social evening". There were a large number of geometrical and astronomical models, as well as film strips, books and calculating machines, being shown by twelve members.

19th March, 1956. Prof. Lighthill again visited the Society to give one of his beautifully clear and dramatic lectures, this time on "Drift". He referred briefly to the theory of secondary flow in pipes and rivers and to the displacement effect for pitot tubes. He then described how secondary flows such as that responsible for the displacement effect could be studied by calculating the drift of the vortex lines in flow about the body. He concluded with some conjectures about changes in this phenomenon to be expected in supersonic flow.

7th May, 1956. The Treasurer's interim report was put before the Annual General Meeting and was discussed. The President announced the resignation of two officers on the Council, Mr. Sowerby, the Vice-President, and Dr. Jacobs, the Secretary. He announced the names of new officers proposed by Council, and three further committee members were elected. After the business meeting, the President, Mr. D. Temple Roberts, delivered his address on "The Dodecahedron". He showed how the building up of a regular dodecahedron leads from the elementary geometry of the pentagon to progressions and recurrence relations—giving scope for the introduction of this subject at various levels in the grammar school. The actual construction was also discussed, and some beautiful models were shown, many made by pupils, in various materials and by various methods. The President also referred to historical connections both in ancient Egypt and Greece, as well as the curious Roman bronze dodecahedra that have been found.

Council for the Session 1956–1957

At the Annual General Meeting the following members were elected to serve on the Council during the next session :

President, Mr. A. Young ; *Vice-President*, Rev. R. A. S. Martineau ; *Treasurer*, Mr. E. J. Watson ; *Secretary*, Dr. G. R. Baldock ; *Committee*, Mr. R. Bromley, Dr. A. Fletcher, Miss L. Harvey, Mr. W. O. McDonald, Mr. D. Temple Roberts (*ex officio*), Miss D. B. Walker, Dr. T. J. Willmore.

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